last update: 13 May 2020

Compound-Angle Joinery When Component Parts Have Unequal Slopes

D. Snyder 21 April 2020

Introduction

This note extends *Compound-Angle Joinery* by D. Snyder and W. Gottesman[1] to include the possibility that the two joined components can have differing slopes or splay angles. The goal is to develop the equations that specify the setup angles for cutting the two components so they meet in either a miter joint or a butt joint, where the setup angles can be used for adjusting the blade and miter-gauge angles of a table saw or other cutting tool used to prepare the components for joining. This extension of [1] is motivated by an e-mail received in April 2020 from Mario Petrilli, who asked about setup angles when splay angles are different.
Equation numbers refer to those in *Compound-Angle Joinery*.

The notation and development used in [1] for developing the setup angles for equal slope angles will be followed in developing the extension to accommodate unequal slope angles. The two components being joined are flat or planer, such as two boards; they are labeled Component 1 and Component 2. References to an equation appearing in [1], for example the equation numbered (n), will be indicated with the notation: Equation (1.n). The slope or splay of a component is the angle in degrees measured from the horizontal to the surface of the component, with counterclockwise direction being positive, as in a right-hand coordinate system. As shown in Fig. 1, S_1 is the slope of component-1, S_2 is the slope of component-2, and θ is the angle between the components, measured in the horizontal plane.

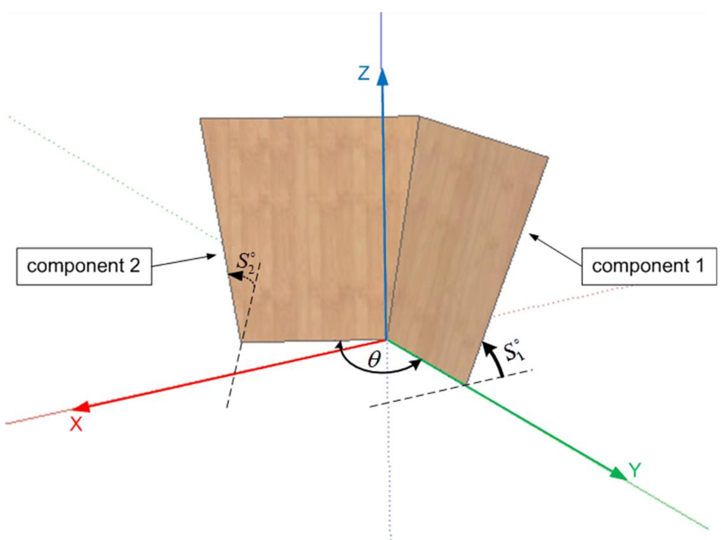

Figure 1. Compound-angle joint connecting two components having unequal slopes

[1] *Compound Angle Joinery*, Donald Snyder and William Gottesman, 2015, available here: http://dls-website.com/documents/WoodworkingNotes/CompoundAngleJoinery.pdf

<u>modification for blade tilt-angle</u>

Equation (1.2) becomes:

$$\vec{u}_1 = R_y\left(S_1\right)\vec{e}_z = \begin{bmatrix} \sin\left(S_1\right) \\ 0 \\ \cos\left(S_1\right) \end{bmatrix} \tag{1}$$

Equation (1.3)[2] becomes:

$$\begin{aligned} \vec{u}_2 &= R_z\left(180^\circ - \theta\right) R_y\left(S_2 - S_1\right)\vec{u}_1 \\ &= \begin{bmatrix} -\cos\left(\theta\right)\sin\left(S_2\right) \\ \sin\left(\theta\right)\sin\left(S_2\right) \\ \cos\left(S_2\right) \end{bmatrix}. \end{aligned} \tag{2}$$

Recognizing that $\vec{u}_1$ and $\vec{u}_2$ are unit vectors, Equation (1.5) becomes $\cos\left(\theta_{dihe}\right) = \vec{u}_1 \bullet \vec{u}_2$, yielding:

$$\boxed{\cos\left(\theta_{dihe}\right) = \cos\left(\theta\right)\sin\left(S_1\right)\sin\left(S_2\right) - \cos\left(S_1\right)\cos\left(S_2\right).} \tag{3}$$

This determines the dihedral angle. The blade angle for a miter joint is $BA^\circ = \theta_{dihe} / 2$, and for a butt joint is θ_{dihe}.

<u>Example 1</u>. equal slopes

If $S_1 = S_2 = S$, then the last equation is identical to Equations (1.5) and (1.10).

<u>Example 2</u>. unequal slopes

If $S_1 = 50^\circ$, $S_2 = 70^\circ$ and $\theta = 90^\circ$, then

$$\cos\left(\theta_{cihe}\right) = -\cos\left(50^\circ\right)\cos\left(70^\circ\right) \approx -0.2198.$$

Then, the dihedral angle is

[2] Equation (1.*n*) refers to equation numbered *n* in Reference [1].

$$\theta_{dihe} = \arccos(-0.2198) = 102.7^\circ.$$

The blade angle for a miter joint is $BA^\circ_{miter} = \theta_{dihe} / 2 = 51.35^\circ$, and for a butt joint it is $BA^\circ_{butt} = \theta_{dihe} = 102.7^\circ$. The blade tilt (saw-blade angle from vertical) is $BT^\circ = BA^\circ - 90^\circ = 12.7^\circ$ for a butt joint, meaning a 12.7° blade tilt counterclockwise from vertical for a table saw. For a miter joint, $BT^\circ = BA^\circ - 90^\circ = -38.65^\circ$, meaning a 38.65° blade tilt clockwise from vertical for a table saw.

modification for miter-gauge angle

For component-1, Equation (1.13) becomes

$$\vec{u}_1 \times \vec{u}_2 = \begin{bmatrix} \sin(S_1) \\ 0 \\ \cos(S_1) \end{bmatrix} \times \begin{bmatrix} -\cos(\theta)\sin(S_2) \\ \sin(\theta)\sin(S_2) \\ \cos(S_2) \end{bmatrix}. \tag{4}$$

Performing the cross product the yields

$$\vec{u}_1 \times \vec{u}_2 = \begin{bmatrix} -\sin(\theta)\sin(S_2)\cos(S_1) \\ -\cos(\theta)\sin(S_2)\cos(S_1) - \sin(S_1)\cos(S_2) \\ \sin(\theta)\sin(S_1)\sin(S_2) \end{bmatrix}. \tag{5}$$

This is the same as Equation (1.13) when $S_1 = S_2 = S$.

Equation (1.14) becomes, for component 1, $\vec{w} = R_y(-S_1)[\vec{u}_1 \times \vec{u}_2]$ or

$$\vec{w} = \begin{bmatrix} \cos(S_1) & 0 & -\sin(S_1) \\ 0 & 1 & 0 \\ \sin(S_1) & 0 & \cos(S_1) \end{bmatrix} \begin{bmatrix} -\sin(\theta)\sin(S_2)\cos(S_1) \\ -\cos(\theta)\sin(S_2)\cos(S_1) - \sin(S_1)\cos(S_2) \\ \sin(\theta)\sin(S_1)\sin(S_2) \end{bmatrix}. \tag{6}$$

This yields

$$\vec{w}=\begin{bmatrix} -\sin(\theta)\sin(S_2) \\ -\cos(\theta)\sin(S_2)\cos(S_1)-\sin(S_1)\cos(S_2) \\ 0 \end{bmatrix}. \tag{7}$$

Equation (1.15) becomes

$$R_z(\varphi_{z1})\vec{w}=\begin{bmatrix} \cos(\varphi_{z1}) & -\sin(\varphi_{z1}) & 0 \\ \sin(\varphi_{z1}) & \cos(\varphi_{z1}) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\sin(\theta)\sin(S_2) \\ -\cos(\theta)\sin(S_2)\cos(S_1)-\sin(S_1)\cos(S_2) \\ 0 \end{bmatrix} \tag{8}$$

This yields a vector $\vec{v}$,

$$R_z(\varphi_{z1})\vec{w}=\vec{v}=\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \tag{9}$$

with

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}=\begin{bmatrix} -\cos(\varphi_{z1})\sin(\theta)\sin(S_2)+\sin(\varphi_{z1})\{\cos(\theta)\sin(S_2)\cos(S_1)+\sin(S_1)\cos(S_2)\} \\ -\sin(\varphi_{z1})\sin(\theta)\sin(S_2)-\cos(\varphi_{z1})\{\cos(\theta)\sin(S_2)\cos(S_1)+\sin(S_1)\cos(S_2)\} \\ 0 \end{bmatrix}.$$

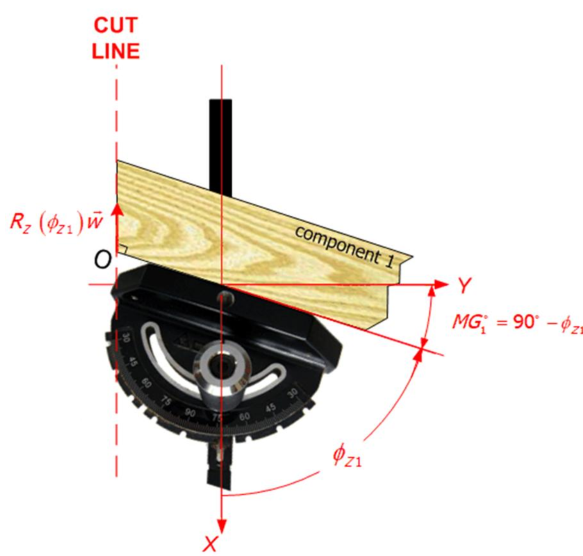

Figure 2. Rotation of Component-1 for cutting

As shown in Fig. 2, the angle φ_{z1} is to be adjusted so that $\vec{w}$ is aligned with the cut line for Component-1. With this alignment, the y component of $\vec{v}$ is zero. Setting $v_y = 0$ yields

$$\tan(\varphi_{z1}) = -\frac{\cos(\theta)\cos(S_1)\sin(S_2) + \sin(S_1)\cos(S_2)}{\sin(\theta)\sin(S_2)}. \tag{10}$$

As indicated in Fig. 2, $MG_1^\circ = 90^\circ - \varphi_{z1}$ is the miter-gauge angle for this alignment. Then,

$$\tan\left(MG_1^\circ\right) = \tan\left(90^\circ - \varphi_{z1}\right) = \frac{1}{\tan(\varphi_{z1})}$$

Thus,

$$\boxed{\tan\left(MG_1^\circ\right) = -\frac{\cos(\theta)\cos(S_1)\sin(S_2) + \sin(S_1)\cos(S_2)}{\sin(\theta)\sin(S_2)}.} \tag{11}$$

A similar sequence of steps for component 2 yields

$$\boxed{\tan\left(MG_2^\circ\right) = -\frac{\cos(\theta)\cos(S_2)\sin(S_1) + \sin(S_2)\cos(S_1)}{\sin(\theta)\sin(S_1)}.} \tag{12}$$

Example 3. unequal slopes

If $S_1 = 50^\circ$, $S_2 = 70^\circ$ and $\theta = 90^\circ$, then $MG_1^\circ = -15.58^\circ$, $MG_2^\circ = -38.25^\circ$; the negative miter-gauge angles means that the miter gauge is rotated clockwise. The blade tilt for either a butt or miter joint is in Example 2.

Bread-Holder Project

This note to extend [1] was motivated by an e-mail received from Mario Petrilli, who asked about setup angles for compound-angle joinery when components have different splay angles. His project involved making a tray to hold a loaf of bread. The four-sided tray was to have two long sides and two short sides, with the long sides having a slope of 70° and the short sides a slope of 50°. The dimensions of the long and short sides are $18 \times 3 \times 0.375$ and $6 \times 6 \times 0.375$ inches, respectively. The sides come together at their bottoms at 90°. Mario planned to use butt joints at the four corners. Four approaches can be used to arrive at setup angles for cutting the component parts.

1. The first method is to use Equations (3), (11) and (12). The blade angle and miter-gauge angles obtained this way are in Example 2 and Example 3.

2. The second method is to follow Mario Petrilli's approach. He used a mixture of 'math' and 'no math' to make a setup block to measure the blade-tilt angle. This complements the strictly 'no math' method described by Steve Brown [2]. Mario describes his method as follows: (12). The angles are as in Example 3: $\theta = 90^\circ$, $S_1 = 50^\circ$ and $S_2 = 70^\circ$.

   "*Take a piece of 2x4, about a foot long, smooth on all sides, with miter gauge at zero (perpendicular) cross cut (wide face down) one end of the block at one of the splay angles, say S1. Now, change the blade tilt to S2 angle. Rip the left edge of the block (face down). We now have a mini-model or template, the side is angled at S2 and the end at S1. Set the blade tilt angle this way. On the right side of the blade, lay the block down on its beveled edge. Set the miter gauge based on the formula values, and set the block against it. Slide the block towards the blade and adjust the tilt on the blade so that it is flush with the beveled end of the block. This is now the blade tilt angle. In my particular case, using a digital gauge, the tilt angle measures 12.7* [degrees]."

3. The third method uses the computer program *SketchUp* to determine setup angles. Shown in Fig. 3 is a rendering in SketchUp of Components 1 and 2. The angles are as in Example 3: $\theta = 90^\circ$, $S_1 = 50^\circ$ and $S_2 = 70^\circ$. By using the procedure of Joe Zeh [3], the miter-gauge angle MG_1° is measured to be 12.7°.

4. A fourth method is to use an interactive, online calculator that is available [4].

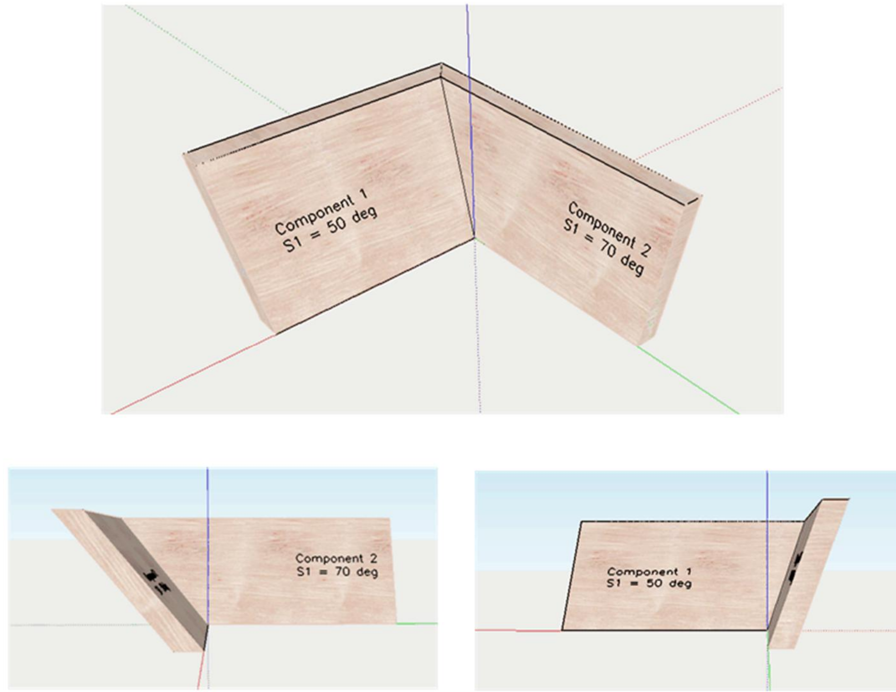

Figure 3. Three views of a SketchUp rendering of Components 1 and 2.

Mario Petrilli's completed bread-holder project

Thanks to Mario for providing pictures of his completed project. The wood is mun ebony.

Figure 4. Mario Petrilli's bread-holder project.
Long sides slope at 70 degrees from horizontal, and the ends slope at 50 degrees.

References

1. *Compound Angle Joinery*, Donald Snyder and William Gottesman, 2015, available here: http://dls-website.com/documents/WoodworkingNotes/CompoundAngleJoinery.pdf

2. Steve Brown, "Compound Angles Without Math," *Fine Woodworking*, pp. 64-67, Sept./Oct. 2002.

3. Joe Zeh, *Compound Miters,* a video lesson using SketchUp, see https://www.youtube.com/watch?v=fG3nz1fjMaI

4. Interactive, online calculator for compound-angle joinery for components having unequal slopes: http://jansson.us/jcompound.html.

Additional websites containing related material

1. https://www.woodweb.com/knowledge_base/Bevel_Cuts_for_SlopeSided_Boxes.html

2. https://www.woodweb.com/knowledge_base/_Spreadsheet_Calculation_Program.html

3. http://pdxtex.com/canoe/compound.htm