Spirals in Woodworking

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Introduction

A two-dimensional spiral is drawn on a flat piece of paper by moving the tip of a pencil in a continuous path that grows smaller or larger as the pencil's tip revolves around a central point. Some examples of two-dimensional spirals are displayed in Fig. 1.[1] Two of interest in woodworking are the Archimedean and Bernoulli spirals, and we will be discussing these in particular.

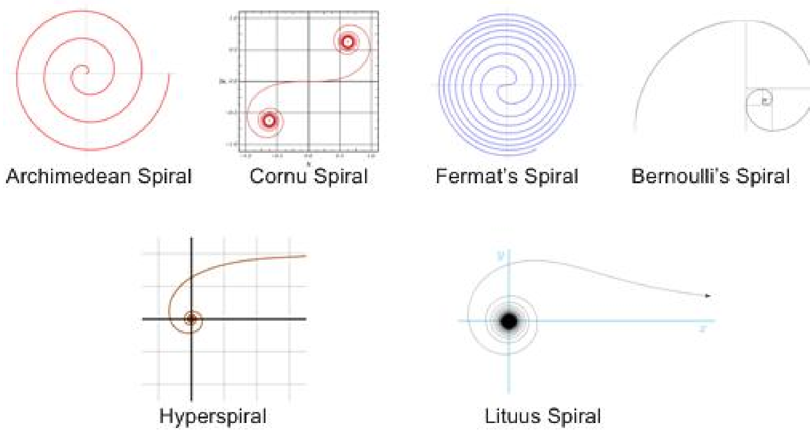

Figure 1. Examples of two-dimensional spirals

Three-dimensional spirals also occur in woodworking. Examples are in Fig. 2.

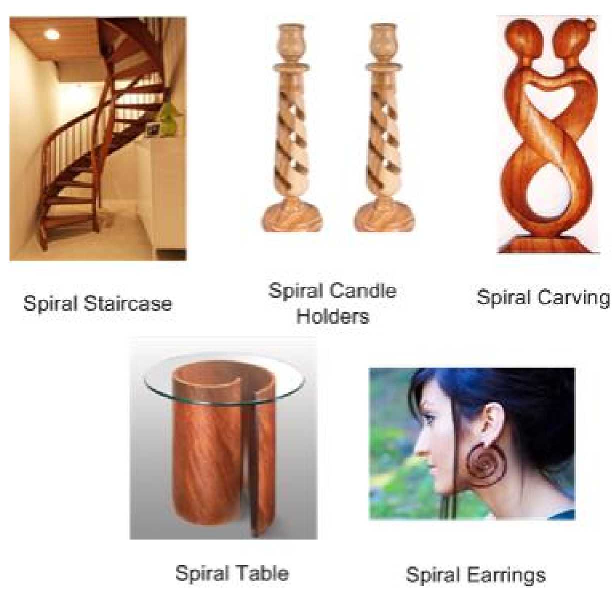

Figure 2. Examples of three-dimensional spirals

As indicated in Fig. 3, a point, p, on a two-dimensional spiral can be characterized by its distance, R, from a central point and the angle, A, that a line between the central point and p makes with a reference direction (here the x axis).

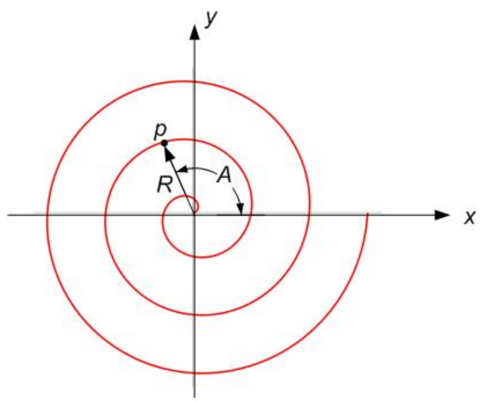

Figure 3. Specifying the location of a point, *p*, on a spiral by its distance, *R*, from a central point and the angle, *A*, between a line to p and a reference (the positive *x* axis)

Alternatively, the location of p can be characterized by its coordinates, x and y, in a Cartesian reference frame. These alternative characterizations of the location of p are related by $x = R\cos A$ and $y = R\sin A$ or, alternatively, by $R = \sqrt{x^2 + y^2}$ and $A = \arctan(y/x)$.

Archimedean Spiral [2]

The distance R and angle A for any point, p, on an Archimedean spiral are related by

$$R = a + bA, \qquad (1)$$

where a and b are constants. The spiral in Fig. 3 is an Archimedean type with $a = 0$ and $b > 0$. The parameter a is the location along the x-axis where the spiral starts (that is, where $A = 0$), and b influences how rapidly the spiral expands ($b > 0$) or contracts ($b < 0$) as the angle A increases. The distance of points on the spiral vary linearly with the angle A. One property of an Archimedean spiral is that successive points on the spiral intercepted by any radial line drawn from the central point are equidistant from one another; for example, any radial line intercepts successive turnings of the spiral at points separated by a distance of $2\pi b$. The tape on a roll of masking tape forms an Archimedean spiral provided the tape has a uniform thickness and is tightly wrapped around a cylinder.

Bernoulli Spiral [3, 4]

The distance R and angle A for any point, p, on a Bernoulli spiral are related by

$$R = c\exp(dA), \qquad (2)$$

where c and d are constants. The spiral in Fig. 4 is a Bernoulli spiral with $c > 0$ and $d > 0$.

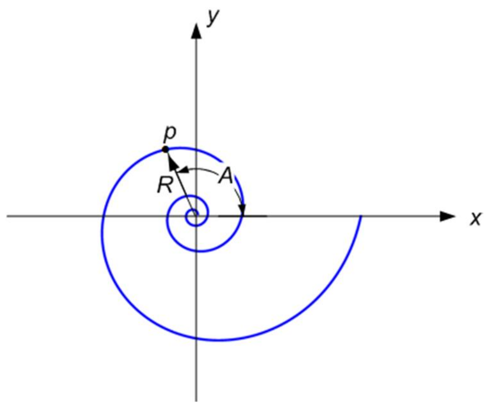

Figure 4. Bernoulli spiral

The distance, R, between a point, p, on the spiral and the central point varies exponentially with the angle A; this distance grows exponentially if $d > 0$ and shrinks exponentially if $d < 0$. So they can be compared, a 360 degree turn of an Archimedean (red) and a Bernoulli (blue) spiral are displayed together in Fig. 5. Parameters for these two spirals are adjusted so they have the same start value at

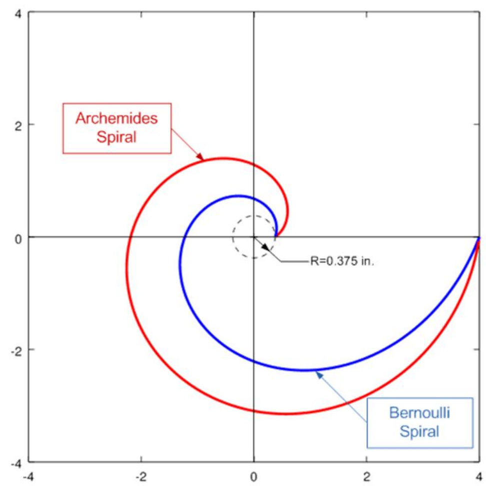

Figure 5. Comparison of Archimedean and Bernoulli spirals

$A = 0^\circ$ of 0.375 in. and same ending value at $A = 360^\circ$ of 4.0 in. For these start and end values, the parameters for the Archimedean spiral are $a = 0.375$ and $b = (4 - 0.375)/2\pi \approx 0.577$, and for the Bernoulli spiral they are $c = 0.375$ and $d = \dfrac{1}{2\pi}\log(4/0.375) \approx 0.377$. Except for their sizes, the two spirals have a similar appearance, but that is misleading because they have quite different properties, properties that make the Bernoulli spiral the more interesting for woodworking applications.

One property of the Bernoulli spiral that makes it interesting for woodworking is its relationship to the "golden ratio," often denoted by the symbol ϕ and having the value $\left(\sqrt{5}-1\right)/2 \approx 1.618$. This ratio is used by some furniture designers as they determine the size relationships between parts of a piece of furniture whether it is a chair, a cabinet, a table or something else. Seth Stem discusses this on pages 94-97 and 103-105 of his book about furniture design [5]. Shown in Fig. 6 are five copies of the Bernoulli spiral of Fig. 4. Rectangles having sides passing through extreme points of the spiral have been drawn on each; the sides are not tangent to the spiral. The ratio of the lengths of the long side to the short side equal ϕ, the golden ratio, for each of these rectangles and any similarly constructed rectangles on any additional turnings of the spiral that might be added; see [7] for a detailed discussion of this property.

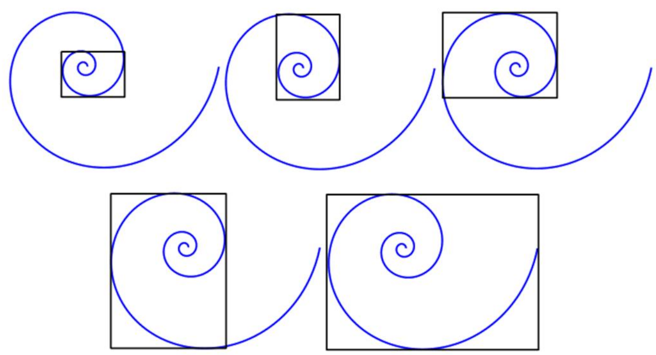

Figure 6. Golden rectangles on a Bernoulli spiral

A Bernoulli spiral also has an "equal angle" property that makes it interesting for woodworking. The angle between a line extending from the central point of the spiral to any point on the spiral and the line that is tangent at that point is the same for all points on the spiral. This equal-angle property is illustrated in Fig. 7. This angle, denoted in the figure by θ, is determined by the parameter d of the spiral according to $\tan\theta = 1/d$.

Consider a radial line drawn in any direction from the central point of an Archimedean spiral, defined in equation (1), and an analogous radial line for a Bernoulli spiral, defined in equation (2). The radial line intersects successive turnings of the Archimedean spiral at points that are equidistant, with the distance being $2\pi b$ for a radial line in any direction and for any pair of turnings. The angle between the tangent at any of these intersection points and the radial line varies from point to point along the spiral. For the Bernoulli spiral, the distance between successive intersection points is not the same, that distance equaling $\left(e^{2\pi d}-1\right)ce^{dA}$. This distance for a Bernoulli spiral varies exponentially with the turning angle A. On the other hand, for a Bernoulli spiral the angle between the tangent and the radial line is the same for all intersection points.

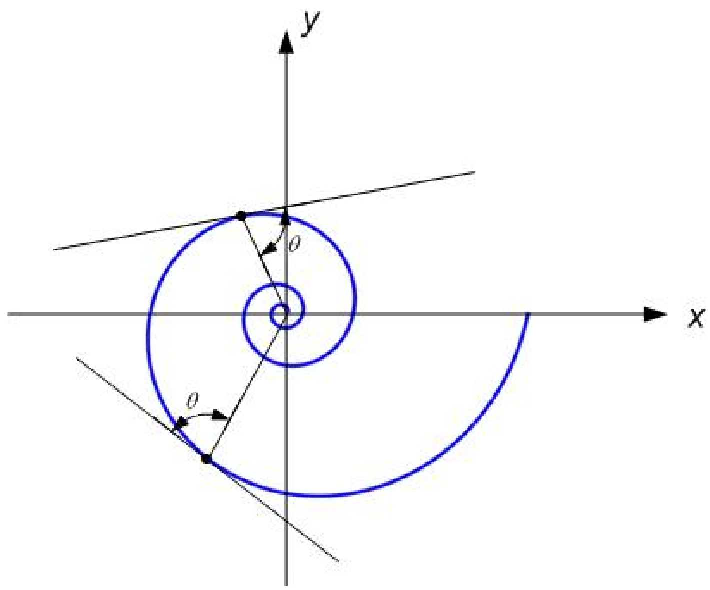

Figure 7. Equal-angle property of Bernoulli spirals

Bernoulli Cam Clamps

Cam clamps are useful additions to woodworking workbenches. They are handy for holding wood pieces on a workbench when smoothing, leveling, or sizing them with a hand plane or shaping them with a chisel. Shown in Fig. 8 is a pair of cam clamps derived from the single turn of a Bernoulli spiral that is in Fig. 5. Fig. 9 shows these clamps in use for surfacing a piece of wood.

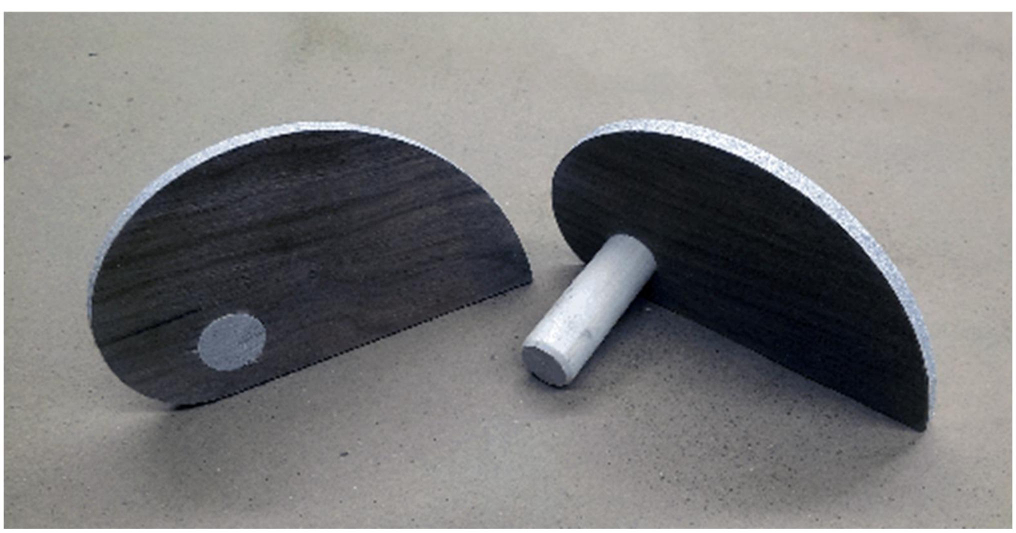

Figure 8. Bernoulli cam clamps

Figure 9. Bernoulli cam clamps in use with a hand plane

In Fig. 9, the two cam clamps grip the wood stock at different positions along the arc of the spirals. This is possible due to the equal-angle property of Bernoulli spirals. The forces on the clamps and wood stock as the hand plane is pushed forward are tangential to the cams. The equal-angle property causes those forces on the clamps and counter forces on the wood stock to be of equal magnitude and direction, so the wood stock is not caused to twist one way or the other as the hand plane is pushed forward. The cam clamps derived from Fig. 5 are made by drilling a hole, centered at the center point of the spiral, for holding a dowel chosen to fit into the bench's dog holes and by cutting the cam material along a line between the terminal point of the spiral and a point of tangency along the spiral. Increasing friction along the gripping edge of the cam is achieved by gluing sandpaper to the spiral edge.

Instructions are attached for making Bernoulli cam clamps. A video instruction is available in Reference [6]

Final Comments

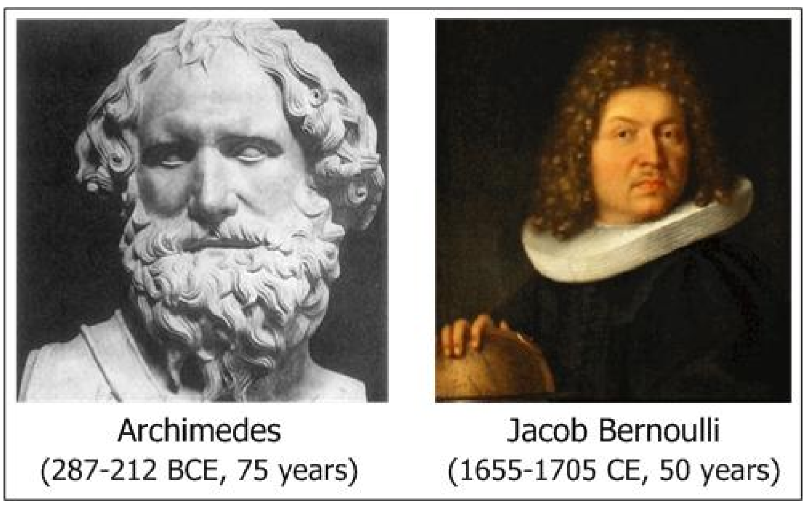

Figure 10. Archimedes and J. Bernoulli

About 2000 years separated the lives of the two mathematicians, Archimedes (287-212 BCE, 75 years) and Jacob Bernoulli (1655-1705 CE, 50 years), whose spirals are discussed above. They come together in a surprising way at the end of Bernoulli's life. Shown in Fig. 11 is a picture of Jacob Bernoulli's gravestone, located in Basel, Switzerland [8]; see [8] for a translation of the Latin inscription into English. Bernoulli wished to have the spiral he studied inscribed on his gravestone, and a spiral is seen at its bottom. However, the spiral displayed appears to have nearly equally spaced turnings, so it is similar more to an Archimedean spiral than to the one studied so much by Bernoulli.

Figure 11. Jacob Bernoulli's gravestone

References

1. https://en.wikipedia.org/wiki/Spiral
2. https://en.wikipedia.org/wiki/Archimedean_spiral
3. https://en.wikipedia.org/wiki/Logarithmic_spiral
4. http://www.2dcurves.com/spiral/spirallo.html
5. Seth Stem, *Designing Furniture: from Concept to Shop Drawing*, Taunton Press, 1989.
6. http://www.garagewoodworks.com/video.php?video=v39
7. http://jwilson.coe.uga.edu/emt668/emat6680.f99/erbas/kursatgeometrypro/golden%20rectangle/goldenrec%26logspiral
8. https://en.wikipedia.org/wiki/Jacob_Bernoulli

Note added on Oct. 20, 2018

The commercial availability of a new style of featherboard appeared on page 16 of Issue No. 269 (Aug. 2018) of *Fine Woodworking Magazine*; see https://www.featherboards.com/. It is called "the Hedgehog." Here is a picture of it:

Figure 12. The Hedgehog featherboard is shaped as a Bernoulli spiral.