ELLIPSES IN WOODWORKING

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I'll begin with an *Introduction* in which there are displayed some ellipses, an oval and some wooden objects having ellipses as an important design element. Then, in *Section 2*, the mathematical description of ellipses is given and some of their properties developed. *Sections 3 and 4* describe some ways to draw or layout ellipses and some ways to cut out elliptical parts for woodworking (and other) projects. *Section 5* is about cove moldings that have segments of ellipses in their design. *Section 6* contains reference citations.

1 INTRODUCTION

An *ellipse* is a 'flattened circle' that is symmetric either side of two lines, one line along the centers of its wide side and the other along the centers of its narrow side. Some ellipses are displayed in Fig. 1.

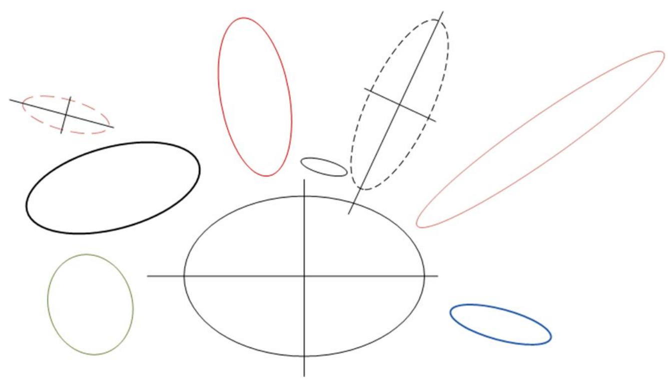


Figure 1. ellipses

A circle is an ellipse with long and short sides of equal length. An oval is a distorted ellipse that is 'egg shaped,' as in Fig. 2. It is symmetric either side of one line along the centers of its wide side.

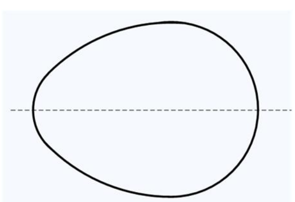


Figure 2. oval

Ellipses or partial ellipses are present in many woodworking projects. Shown in Fig. 3 are several examples.

Figure 3. Examples of ellipses and ovals in woodworking. (a) elliptical foot stool, (b) oval cutting board, (c) elliptical cutting board, (d) elliptical table-top, (e) Shaker box, (f) elliptical mirror, (g) elliptical window frame, (h) trim molding, (i) framing for an elliptical ceiling feature, (j) elliptical archway, (k) demilune table, (l) elliptical serving tray, (m) bracelet of elliptical parts.

The mathematical definition and some properties of ellipses are discussed next. Then some practical aspects of ellipses are reviewed; this includes laying out or drawing ellipses on materials for cutting or other purposes and some methods for performing the cutting.

2 DEFINITION AND PROPERTIES OF ELLIPSES

A standard ellipse is a closed curve in a plane, as shown in Figure 1. It has major (-M,M) and minor (-m,m) axes aligned with the axes of a Cartesian coordinate system, with its center, C , located at the origin

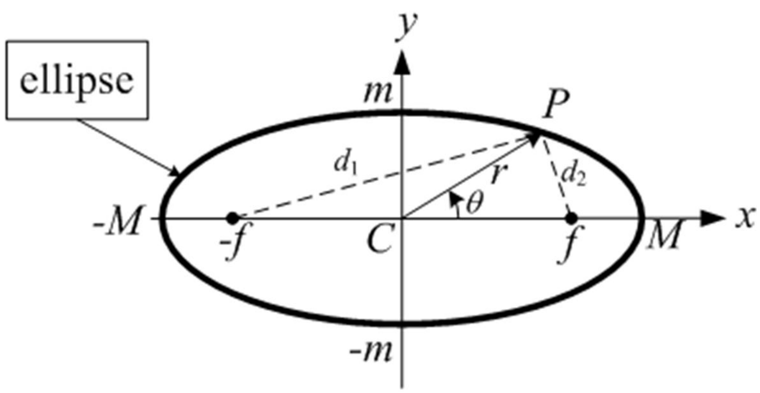


Figure 1. A standard ellipse

$C=(0,0)$ of the coordinate system and its long and short axes aligned with the *x*-axis and *y*-axis, respectively, of the coordinate system.[1] The equation for points $P(x,y)$ that lie on the elliptical curve is

$$\frac{x^2}{M^2}+\frac{y^2}{m^2}=1 \tag{1}$$

where M and m are parameters that determine the size of the major and minor axes. If M and m are equal, say to *r*, then the ellipse is a circle of radius *r*. If $M>m$, the major axis of the ellipse lies along the x axis of the coordinate system, as shown in Figure 1. An alternative way to represent points $P(x,y)$ that lie on the ellipse is in terms of an angular parameter φ , with $0 \le \varphi \le 360^\circ$. In this representation,

$$(x,y)=(M\cos\varphi, m\sin\varphi). \tag{2}$$

A point $P(x,y)=P(M\cos\varphi, m\sin\varphi)$ lies on the ellipse because

[1] The coordinate system used here is a "right-handed" system with positive angles measured counterclockwise from the positive *x*-axis.

$$\frac{(M\cos q)^2}{M^2}+\frac{(m\sin q)^2}{m^2}=1.$$

As j varies from 0º to 360º, the point *P* moves from $(x,y)=(M,0)$ counterclockwise around the ellipse and back to the starting point. Note that this angle j is *not* the same as the angle q in Fig. 1, and $P(M\cos j,m\sin j)$ is not the same as the point P in Fig. 1 ; a polar representation of points on the ellipse is given below in terms of the polar distance and angle (r,q) in Fig. 1. The center *C* of the standard ellipse lies at the origin (0,0) of the coordinate system. The points (0,–*f*) and (0,+*f*), with $f=\sqrt{M^2-m^2}$, are called the *focal points* of the ellipse. The distance d_1 from the focal point at (0,–*f*) to a point $P(M\cos j,m\sin j)$ on the ellipse is

$$\begin{aligned} d_1(j) &= \sqrt{(f+M\cos j)^2+(m\sin j)^2} \\ &= \sqrt{M^2-m^2+2Mf\cos j+M^2\cos^2 j+m^2\left(1-\cos^2 j\right)} \\ &= \sqrt{M^2+2Mf\cos j+f^2\cos^2 j} \\ &= M+f\cos j\,. \end{aligned} \qquad (3)$$

Similarly, the distance $d_2(j)$ from the focal point at (0,+*f*) to $P(M\cos j,m\sin j)$ is $M-f\cos j$. This yields the important property that for *any* point on the ellipse, the sum of the distances from the two foci to that point is a constant equal to the length, $2M$, of the major axis,

$$d_1(j)+d_2(j)=2M\,. \qquad (4)$$

The *eccentricity* of an ellipse, *e*, is defined by

$$e=\sqrt{1-\frac{m^2}{M^2}}=\frac{f}{M} \qquad (5)$$

The eccentricity has a value between 0 and 1, 0 ≤ *e* ≤ 1. An ellipse having an eccentricity e = 0 is a circle. As illustrated in Figure 2, the ellipse departs more and more from a circle as *e* increases, becoming simply a line when *e* = 1.

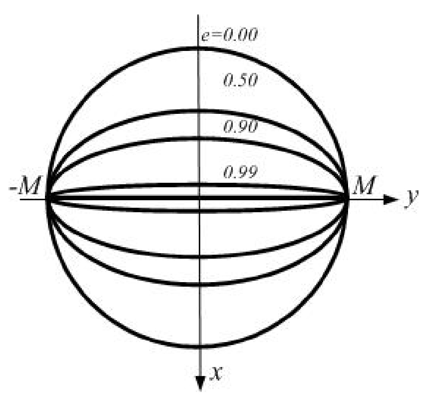


Figure 2. Illustration of the effect of eccentricity on the shape of an ellipse

The distance from the ellipse center, at $(0,0)$, to a point on the ellipse, $r = r(q)$ in Fig. 1, can be expressed in terms of the eccentricity and polar angle q as follows. The location (x, y) of a point $P(x, y)$ on the ellipse is given in terms of the point's polar coordinates (r, q) by $(r\cos q, r\sin q)$. Equation (1) becomes

$$\frac{r^2\cos^2 q}{M^2} + \frac{r^2\sin^2 q}{m^2} = 1. \qquad (6)$$

Solving for r as a function of q and doing some manipulations yields

$$r(q) = \frac{m}{\sqrt{1 - (e\cos q)^2}} \; . \qquad (7)$$

So, as q varies from 0° to 360°, a point at $(r(q), q)$ traces the ellipse.

The equation for an ellipse that is not centered at the origin (0,0) but, rather, shifted so its center is at (x_0, y_0) is

$$\frac{(x - x_0)^2}{M^2} + \frac{(y - y_0)^2}{m^2} = 1. \qquad (8)$$

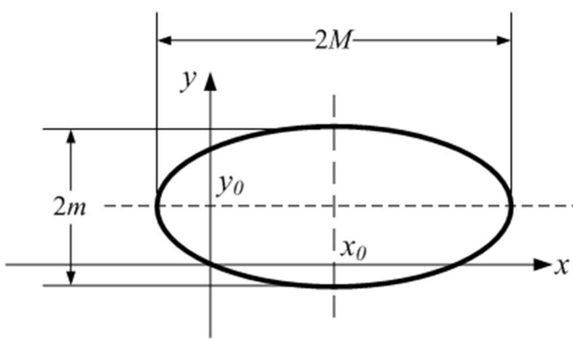


Figure 3. ellipse shifted

The equation for an ellipse that is shifted to the point (x_0, y_0) and rotated by an angle y is

$$\frac{[(x - x_0)\cos y + (y - y_0)\sin y]^2}{M^2} + \frac{[-(x - x_0)\sin y + (y - y_0)\cos y]^2}{m^2} = 1. \qquad (9)$$

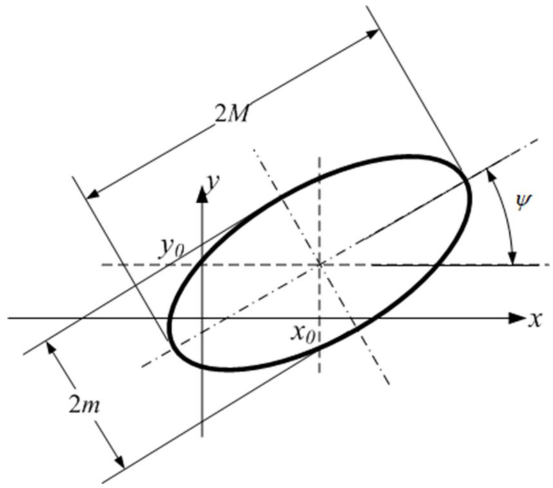


Figure 4. ellipse shifter and rotated

3 LAYING OUT AND DRAWING ELLIPSES

There are a variety of ways to draw or layout ellipses. Following are descriptions of three methods.

a. pin and string method

Assume that a coordinate system with x and y axes is given and that the sizes of the minor and major axes, $2m$ and $2M$ are known. The two focal points are located at $\pm f = \pm\sqrt{M^2 - m^2}$ along the major axis. Place a pin at each of the focal points, with each pin holding an end of an inextensible string of length $2M$. The tip of a pencil is used to pull the string taut and make a mark. As the pencil is moved while keeping the string taut, an ellipse is drawn because of the property expressed in (4).

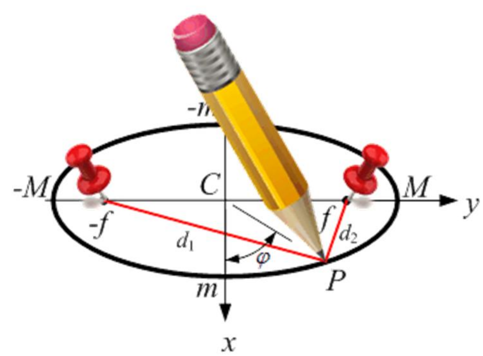


Figure 5. pin and string method for drawing an ellipse

b. trammel method

A trammels is used while drawing or laying-out an ellipse to enforce constraints dependent on the sizes of its major and minor axes, $2M$ and $2m$. Trammels can be made in various ways. Two are shown in Fig. 6. The trammel in Fig. 6(a) is made by drilling three small, properly placed holes in piece of wood or other material, with the holes spaced according to the size of the desired ellipse. An alternative is to use a strip of paper with properly spaced tick marks along an edge. The trammel in Fig. 6(b) is made using trammel points attached to a ruler, with the spacing between points set according to the size of the desired ellipse. Draw orthogonal x and y axes, and place the trammel at any angle j while constraining the center of hole-3 to be on the x-axis and the center of hole-2 on the y-axis, as in Fig. 7. The center of hole-3 then lies on the ellipse because the coordinates of its center are $(x, y) = (m\cos\varphi, M\sin\varphi)$, which is (2).

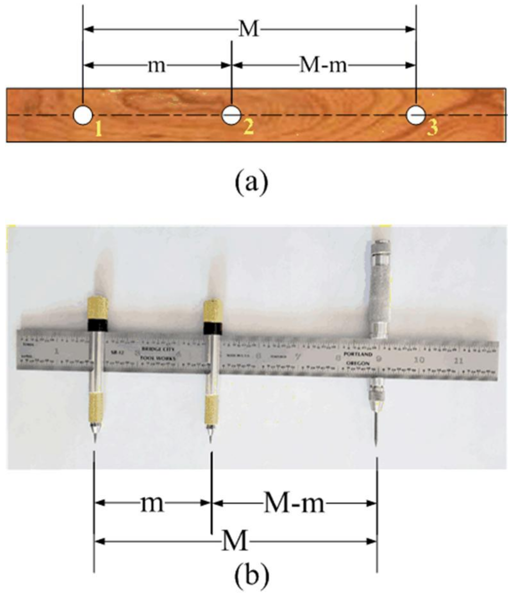


Figure 6. (a) trammel made with three small holes, (b) trammel made using trammel points

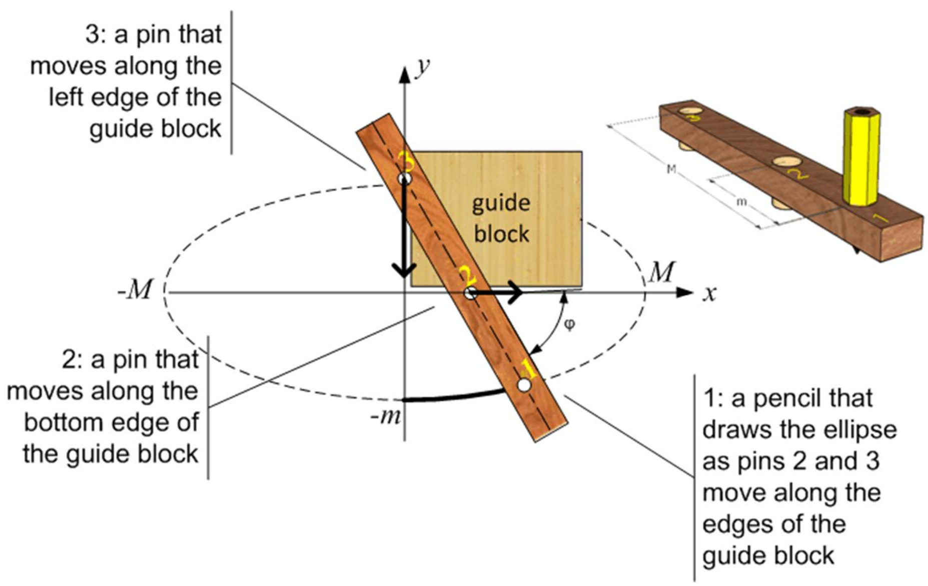


Figure 7. drawing an ellipse using a trammel

A continuous elliptical-curve can be drawn using the pin and pencil method. It is interesting that a continuous elliptical-curve can be accomplished using the trammel method as well. One approach is to place pins in holes 2 and 3 that extend a short distance beneath the bottom surface of the trammel and a pencil in hole 1, as in Fig. 7. The pins are moved smoothly along the edges of a guide block that is aligned with the x and y axes of one quadrant of the coordinate system and offset so that the centers of the pins track along the two axes. This permits one quadrant of the ellipse to be drawn. The other three quadrants can be drawn by placing the guide in those quadrants and repeating the drawing for each quadrant of the ellipse. An alternative that permits continuous drawing of all four quadrants of the ellipse is to construct a guide block with crossing channels, as in Fig. 8, with channels having a width slightly larger than the trammel pins so the pins can move freely along straight lines without binding. This trammel configura-

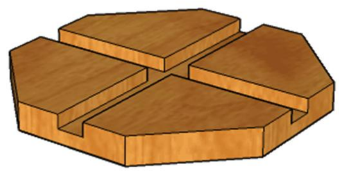
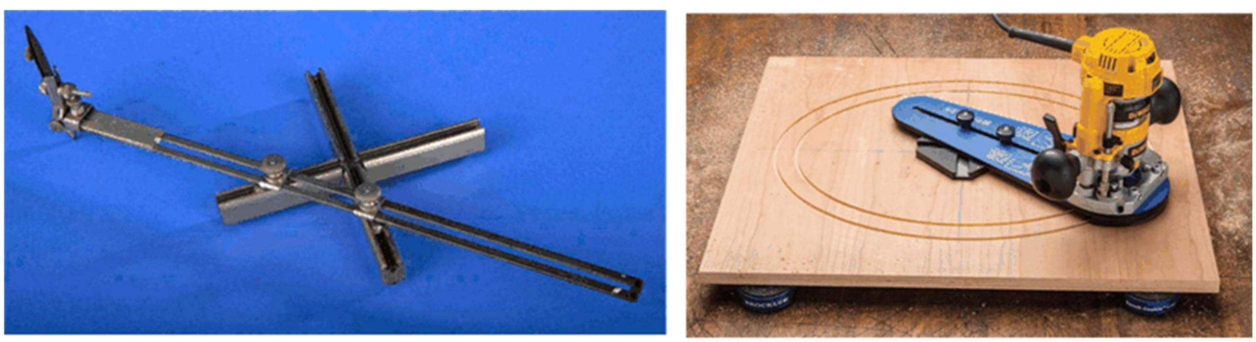


Figure 8. four quadrant trammel-guide

tion for drawing ellipses has been known since the 1600s and is often called the “trammel of Archimedes” [9, 10]. Two ellipse trammels are shown in Fig. 9.



(a) (b)

Figure 9. Ellipse trammels: (a) trammel made by Keuffel and Esser, c. 1930, in the National Museum of American History of the Smithsonian Institution, (b) router guide for cutting ellipses by Rockler Woodworking and Hardware Co.

c. point-by-point methods

It may sometimes be desired to locate discrete points or locations on an ellipse without laying out the ellipse as a continuous curve. This can be accomplished by establishing orthogonal x and y coordinates and then placing marks at desired, predetermined locations (x_n, y_n) on the ellipse for as many locations, $n = 1, 2, \ldots, N$, as desired. Woodworkers are familiar with the use of “story sticks” to record key dimensions in a project and to reuse those sticks when the project is pursued again [15-19]. A *story stick* is just a piece of wood, paper or other material that has multiple marks on it for recording in full scale size a visual record of important dimensions; the marks can be in the form of scribe lines, holes or other indicators. Here is an example. Suppose points are desired on an ellipse having given major and minor axes of sizes M and m aligned on x and y coordinates as in Fig. 1. On a stick somewhat longer than $2m$ mark three locations, one at the center and the other two at a distance m either side of the center. Assume that N desired locations on the ellipse are to be at the precomputed coordinates (x_n, y_n) for $n=1,2,\ldots,N$. The distances from the minor-axis terminus at $y=+m$ to each of these coordinates are

$$\text{distance}\left[(0,m) \rightarrow (x_n, y_n)\right] = \sqrt{x_n^2 + (y_n - m)^2}, \quad n=1,2,\ldots,N .$$

Mark each of these N distances from a reference mark on a second stick. Similarly, the distances from the minor axis terminus at $y=-m$ to each coordinate are

$$\text{distance}\left[(0,-m) \rightarrow (x_n, y_n)\right] = \sqrt{x_n^2 + (y_n + m)^2}, \quad n=1,2,\ldots,N .$$

Mark a third stick with these distances. The three sticks can now be placed to locate the *n*-th point on the ellipse as in Fig. 10, this being accomplished, requiring only that the x and y axes be laid out but not the full, continuous ellipse.

Example Table 1 displays the x and y coordinates for 15 points on an ellipse having major and minor axes of dimensions $M=5$ and $m=3.12$, respectively. It is unnecessary for purposes of this note to know a motivation for these points, but they are special in the design of an *analemmatic sundial* designed for St. Louis, MO, as developed in [20-22]. Shown in Fig. 10 are three story sticks in position

Table 1. 15 points (labeled by hour) on an ellipse with $M=5$ and $m=3.12$

hour	x	y	distance to $(0,-m)$	distance to $(0,+m)$
5	–4.83	–0.81	5.35	6.22
6	–5.00	0.00	5.89	5.89
7	–4.83	+0.81	6.22	5.35
8	–4.33	+1.56	6.38	4.60
9	–3.54	+2.21	6.39	3.65
10	–2.50	+2.70	6.34	2.53
11	–1.29	+3.01	6.27	1.30
12	0.00	+3.12	6.24	0.00
13	+1.29	+3.01	6.27	1.30
14	+2.50	+2.70	6.34	2.53
15	+3.54	+2.21	6.39	3.65
16	+4.33	+1.56	6.38	4.60
17	+4.83	+0.81	6.22	5.35
18	+5.00	0.00	5.89	5.89
19	+4.83	–0.81	5.35	6.22

for marking the point on the ellipse labeled hour 16. A picture of this story-stick method for laying out points on an ellipse is shown in Fig. 11 for an analemmatic sundial being constructed by Brain Albinson [23-25].

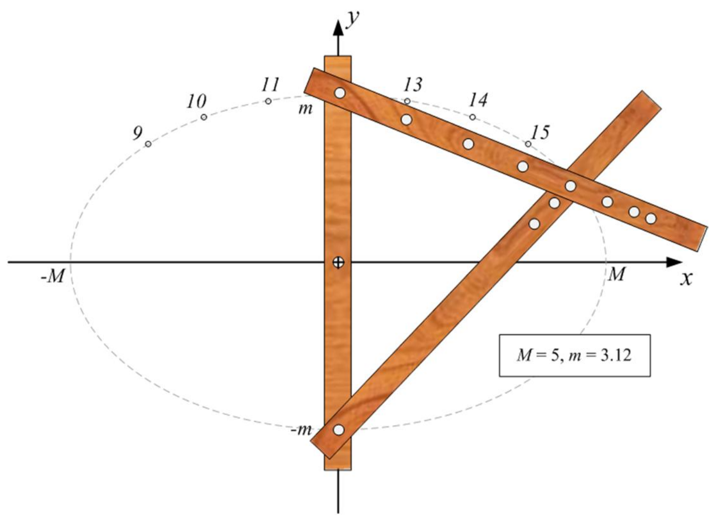


Figure 10. Story sticks arranged to mark the point labeled hour 16 on the ellipse

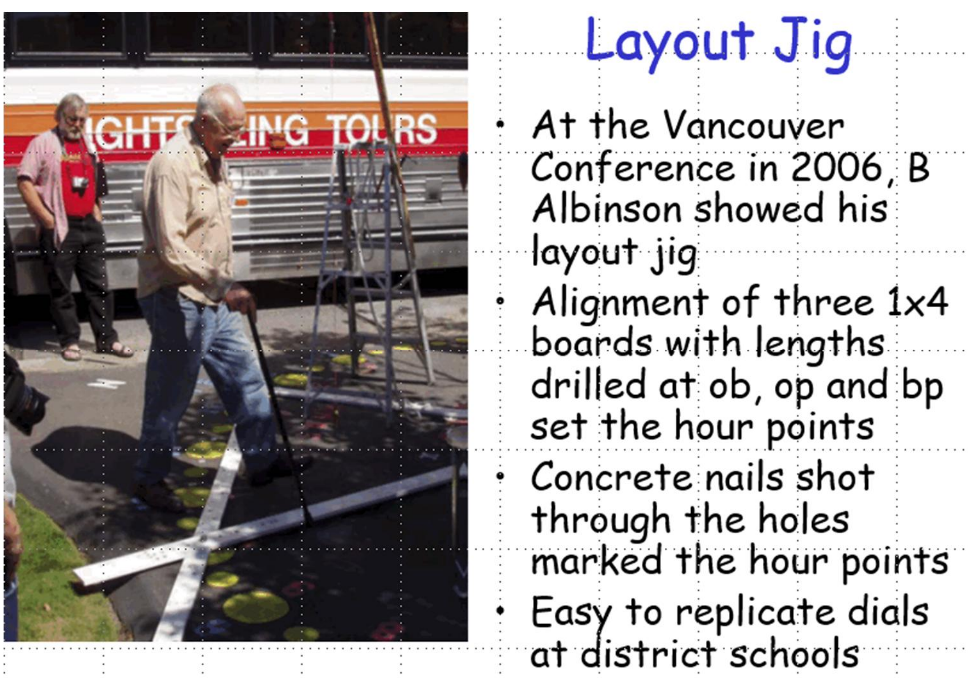


Figure 11. Slide from a presentation by Roger Bailey [23]

4 COVES

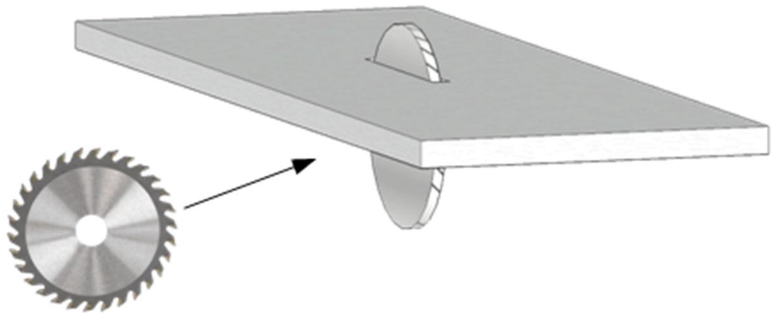


Figure 12. Model of a table saw

A model of a table-saw surface and saw blade is in Fig. 12. When looking at the blade from a direction that is perpendicular to its surface, the tips of the cutters on the blade lie on a circle, but from oblique angles the tips appear to lie on an ellipse. A board passing over the saw blade at an oblique angle results in the upper segment of that ellipse being cut into the board's surface, as in Fig. 13(a). Tilting the sawblade results in a cove that is the upper segment of a rotated ellipse, as in Fig. 13(b).

(a) (b)

Figure 13. Cutting coves having elliptical profiles:
(a) from Woodgears, An Engineer's Approach to Woodworking [29]; (b) from Woodgears [30]

Once the blade diameter, d , is selected, there are three adjustments to be made when setting up the tablesaw to cut a cove: the height, h, of the blade above the surface of the tablesaw; the tilt angle, f , of the saw blade; and the angle, q , between the workpiece and the plane of blade's surface. The cove has two parameters: the width, w, of the cove; and the offset, o, of the peak of the cove from the surface of the tilted blade. These interdependent parameters are indicated in Fig. 14. There are on-line calculators for determining the setup parameters [31-34]. Refs. [31, 34] require d , h, w and o as user-input parameters, and then compute f and q . Refs. [32,33] assume no offset ($o=0$) and require as user-input parameters d , w and h, and then computes q .

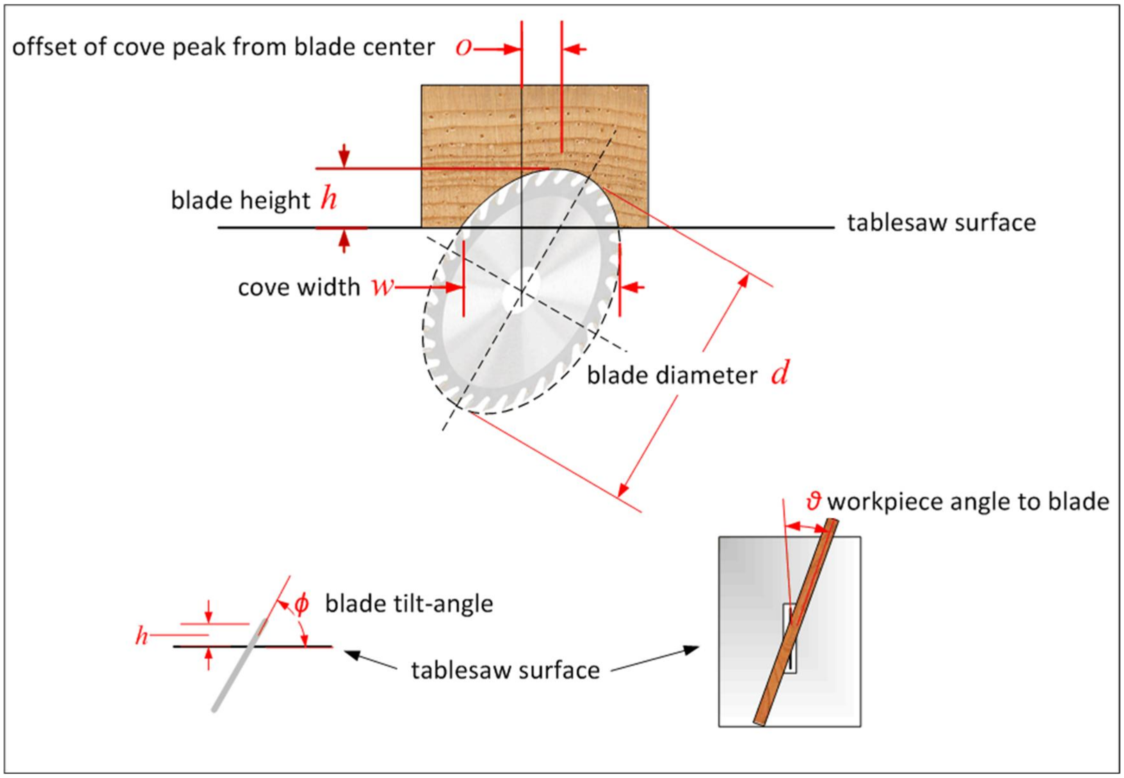


Figure 14. Cove parameters: h, w and o; blade parameters: d, ϕ and θ

In a three-dimensional space with cartesian coordinates x, y and z, the sawblade is circumscribed by a circle in the y, z - plane with radius $r = 0.5d$,

$$\left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = 1. \tag{10}$$

If the sawblade is not tilted but is perpendicular to the tablesaw's surface (i.e., in Fig. 14, $\phi = 90^{\circ}$ and $o = 0$), the ellipse circumscribing the sawblade is this circle parallel projected in the direction θ onto a plane with orthogonal coordinates $(y', z) = (y \sin\theta, z)$ of the workpiece; the equation for this ellipse is

$$\left(\frac{y'}{r \sin\theta}\right)^2 + \left(\frac{z}{r}\right)^2 = 1. \tag{11}$$

A sawblade raised above the tablesaw's surface by a height h, produces a cove having a depth h and a width w that depend on θ. This relationship is obtained by setting $y' = 0.5w$ and $z = r - h = 0.5(d - 2h)$ in (11), yielding

$$0.25\left(\frac{w}{\sin\theta}\right)^2 + 0.25(d - 2h) = r^2 = 0.25d^2 ,$$

which reduces to

$$w = 2\sqrt{h(d - h)}\,\sin\theta \,. \tag{12}$$

This provides a constraining relationship between the parameters w, h, d and θ of Fig. 14. Specifying any three of these parameters then permits the fourth one to be calculated for cutting coves with zero offset $(o = 0)$.

Example. Suppose a cove having no offset $(o = 0)$, a depth $h = 0.5$ and a width $w = 1.0$ inches is desired and that at sawblade having a diameter of $d = 10$ inches is to be used. Then from (12) the angle that the workpiece should make with the plane of the sawblade is

$$\theta = a\sin\left(\frac{w}{2\sqrt{h(d - h)}}\right) = a\sin(0.229) = 13.26^{\circ} \,.$$

Two issues remain. One is setup angles when the desired offset is not zero, yielding an asymmetric cove; see the calculators in Refs. [31-34]. The second is safety; cutting coves on a tablesaw can be hazardous, so adopting safe methods for doing the work is important; see Refs. [27-34] for some discussions about this issue. One aid to safely cutting coves is to use homemade or commercial “parallelogram” jig to guide the workpiece across the sawblade, such as the commercial one shown in Fig. 15.

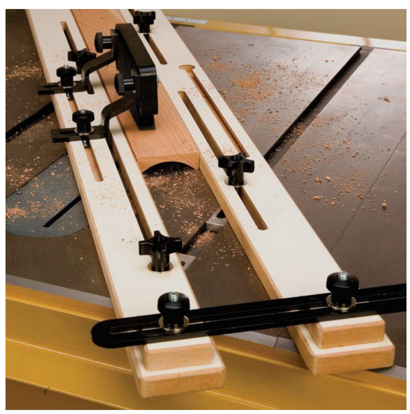


Figure 15. Cove-cutting jig available from Rockler Woodworking and Hardware Co.

5 Ellipses of Another Kind

> ellipsis [I'lip sis] (*noun*) g ellipses (*plural noun*)
>
> *Def*. the omission of words from a sentence or other construction that would complete the construction but are unnecessary for understanding because of contextural clues. A mark as g g g to indicate an omission of letters or words.

More about the mathematics and construction of ellipses can be found in books, magazines and the Internet, some of which will be helpful to woodworkers. The notes above cover some important aspects, but if the reader is interested to explore more g g g .

6 References

There are many discussions of all aspects ellipses in magazines [1-7] and websites [5, 8-10], as a search on the Internet will reveal. Ellipses occur in many woodworking applications; examples include cutting coves [26-28], Shaker boxes [11,12], decorative inlays [13] and furniture [3, 7,14]. They occur as a central element in other applications as well [20-25].

1. Ed Moore, Rufus Winsor, Fred Johnson, C. W. Beringhaus and Steele Hinton, "Drawing the Ellipse: Several ingenious methods," *Fine Woodworking Magazine*, No. 22, pp. 72-73, May/June 1980.
2. Len Schmidt, "Making elliptical mirror frames," *Fine Woodworking Magazine*, No. 86, pp. 88-90, Feb. 1991.
3. Chris Gochnour, "A Table for 2, 4, or 6," *Fine Woodworking Magazine*, No. 255, pp. 30-37, Aug. 1993.
4. Robert Gabor, "Ellipse drawing aid," *Fine Woodworking Magazine*, No. 119, p. 16, Aug. 1996.
5. Frank Fitzpatrick, "A precise method for drawing an ellipse," *Fine Woodworking Magazine*, No. 121, p. 8, Dec. 1996, and https://www.finewoodworking.com/2006/04/11/a-precise-method-for-drawing-an-ellipse.
6. Gary Rogowski, "Three ways to lay out an ellipse," *Fine Woodworking Magazine*, No. 139, p. 112, Dec. 1999.
7. Chris Gochnour, "Make an elliptical tabletop," *Fine Woodworking Magazine*, No. 255, pp. 82-85, Aug. 2014.
8. Bob Miller, "How to draw an ellipse," https://www.finewoodworking.com/2019/10/02/how-to-draw-an-ellipse, Oct. 2019.
9. Ellipse, https://en.wikipedia.org/wiki/Ellipse.
10. Trammel of Archimedes, https://en.wikipedia.org/wiki/Trammel_of_Archimedes#Nothing_grinder.
11. Pete Baxter, "Make a Shaker Box," *Fine Woodworking Magazine*, No. 86, pp. 38-43, May/June 2015.
12. John Wilson, "Shaker Oval Boxes," *Fine Woodworking Magazine*, No. 247, pp. 54-57, Sept./Oct. 1993.

13. Steve Latta, "Federal-Style Oval Inlays," *Fine Woodworking Magazine*, No. 86, pp. 70-75, Sept./Oct. 1999.
14. Randall O'Donnell, "Oval Chippendale Stool," *Fine Woodworking Magazine*, No. 135, pp. 55-61, Mar./Apr. 1999.
15. https://www.canadianwoodworking.com/tipstechniques/story-sticks
16. Matthew Teague, "A light settee in cherry," *Fine Woodworking Magazine*, No. 147, pp. 61-67, Feb. 2001.
17. Scott Gibson, "An everyday cabinet: straightforward construction methods for building a case with doors and drawers," *Fine Woodworking Magazine*, No. 152, pp. 66-73, Dec. 2001.
18. Scott Gibson, "An everyday cabinet: straightforward construction methods for building a case with doors and drawers," *Fine Woodworking Magazine*, No. 152, pp. 66-73, Dec. 2001.
19. Garrett Hack, "Use patterns to guide your work," *Fine Woodworking Magazine*, No. 246, pp. 78-80, April 2015.
20. Donald Snyder, http://dls-website.com/documents/Analemmatic_Sundials.pdf.
21. Carl Sabanski, http://www.mysundial.ca/tsp/analemmatic_sundial.html.
22. https://plus.maths.org/content/os/issue11/features/sundials/index
23. Roger Bailey, "Layout an analemmatic sundial," presented at the 2017 Annual Conference of the North American Sundial Society, St. Louis, MO.
24. Brian Albinson, http://www.mysundial.ca/sotw/sotw_canada_ba.htm
25. Brian Albinson, http://www.sfu.ca/archive-sfunews/sfu_news/archives_2003/sfunews10300310.htm
26. Frank Klausz III, *Cove Cutting: A Guide to Cutting Symmetrical and Asymmetrical Curves Using the Table Saw*, Klausz-Tech Publishing, 1993.
27. Frank Klausz III, "Coves Cut on the Tablesaw," *Fine Woodworking Magazine*, No. 201, pp. 82-85, Sept./Oct. 1993.
28. Will Neptune, "Cutting Coves on the Tablesaw," *Fine Woodworking Magazine*, No. 281, pp. 62-70, May/April 2020.
29. https://woodgears.ca/cove/
30. https://woodgears.ca/cove/asymmetric.html
31. http://thewoodpecker.net/cove/cove.htm
32. https://www.sawdustzone.org/articles/miscellaneous/825089-cove-cut-calculator
33. http://www.shopnotes.com/files/issues/116/cove-calculations.pdf
34. https://www.finewoodworking.com/online-tools/cove-angle-calculator
35. https://math.stackexchange.com/questions/91132/how-to-get-the-limits-of-rotated-ellipse
36. http://quickcalcbasic.com/ellipse%20line%20intersection.pdf
37. https://gamedev.stackexchange.com/questions/159408/find-extreme-points-of-a-rotated-ellipse-function-on-a-given-axis