Compound-Angle Joinery
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1   Introduction

In a previous note, one of us (DLS) wrote about compound-angle joinery and stated, without explanation, the mathematical expressions which specify the blade tilt and miter-gauge setup angles on a table saw used to cut the parts of a compound-angle joint [1]. The same expressions apply for other tools used to cut the joints, such as miter saws, scroll saws, and even hand saws. For completeness, the mathematical basis for the expressions is developed in this note. In addition, the earlier Frink computer-procedure [1] for calculating the setup angles is updated to include correct formulas for setup angles and for not just compound miter-joints but also compound butt-joints, and the visual display of results on portable devices such as smart phones and tablets is improved. The mathematical basis for compound-angle joinery is developed in two ways. For the first, in Section 3, plane trigonometry, vector notation and rotation matrices are used. For the second, in Section 4, spherical trigonometry is used following insights of Bill Gottesman.

In Section 1.1 we give some examples of objects made using compound-angle joinery. Several angles that occur in the joinery are identified in Section 2. The development of the setup angles by using plane trigonometry, vectors and rotation matrices is in Section 3. The development using spherical trigonometry is in Section 4. Results are summarized in Section 5, and computer implementations are in Section 6. Section 7 has four examples, and 8 lists cited references.

1.1   Examples of objects exhibiting compound-angle joinery

Compound-angle joinery is an integral feature of diverse objects. Examples of *closed forms* shown in Figs. 1-4 illustrate the wide range of objects where this joinery is encountered. By a 'closed form' is meant a multisided object in which the multiple sides are connected to form an enclosure, such as a box.

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[1] Bill Gottesman of Burlington, Vermont, contributed the ideas and material for the section in which spherical trigonometry is used to develop expressions for the setup angles needed for compound-angle joinery.

Figure 1. Octagonal jewelry box with removable shelf
insert. Made by Vic Barr using maple and cherry
woods. The eight sides are at 90° to the base.

rectangular trays   heptagon vase   hexagon table-lamp

Figure 2. Four-, seven- and six-sided closed-form objects
involving compound-angle joinery.

Figure 3. Hexagonal bowl involving compound-angle
joinery. Each of the six sides slope outward at 60° from the base.

Figure 4. Four sided sea chest having sides
that slope inward at 82° from the base.

There are also *open forms* that involve compound-angle joinery. These are generally not enclosures but after twists and turns may become closed. Examples are in Fig. 5. Ceiling trim often winds around the ceiling eventually closing on itself like a snake eating its own tail.

Figure 5. Compound-angle joinery in ceiling and fireplace-mantel moldings

The examples given above of closed forms involve multiple compound-angle joints that are identical.   Closed forms that do not have repeated compound-angle joints are also possible. We treat these by considering each joint separately as an open form once the basic shape is selected.

2   Compound joints and cut planes

Compound-angle joints are characterized by two important angles and by planes that define how the parts are cut so they can be joined to form the compound joint.

2.1 Angles that characterize a compound-angle joints

Illustrated in Fig. 6 are two components of a closed or open form that come together in a compound-angle joint. Both components are assumed to rest on the *XY*-plane of a three-dimensional coordinate system[2], with the resting edge of one component (*component 1*) aligned with *Y*-axis and having a slope of *S* degrees measured (counterclockwise) from the *XY*-plane. The origin of the coordinate system is located at the point in the *XY*-plane where the two components come together. The resting edges of the two components meet an angle *q* in the *XY*-plane. For a closed form having the shape of an *N*-sided regular polygon in the *XY*-plane, $q = (N - 2) * 180^0 / N$ degrees. For example, a four-sided square box will have $q = 2 * 180^0 / 4 = 90^0$ regardless of any slope the sides may have, the six-sided bowl of Fig. 3 has an hexagonally shaped base with $q = 4 * 180^0 / 6 = 120^0$, and the seven-sided heptagon-vase of Fig. 2 has $q = 5 * 180^0 / 7 \approx 128.6^0$. Generally, $0 \le q \le 180^0$.

There is another angle that is important in describing the joined components of Fig. 6. It is called the "dihedral angle." The dihedral angle can be measured by constructing two lines, one in the face of each component. The line in a face is constructed so that it is perpendicular to the mating line where the two faces join. The two constructed lines are positioned to meet at common point anywhere along the mating line. The smallest angle between the two lines constructed in this way is the dihedral angle (also called the *plane angle* [11]).
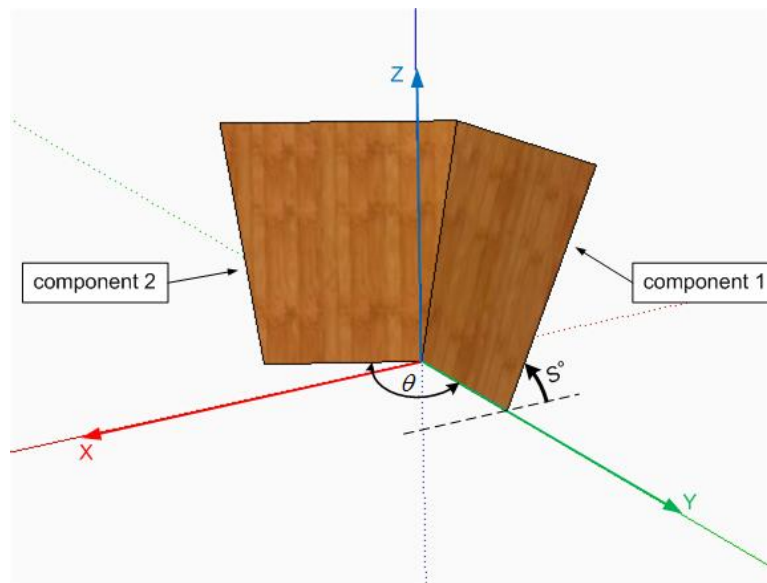

Figure 6. Compound-angle joint connecting two components
of a closed or open form

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[2] A coordinate system with a right-hand convention is used. Angles measured counterclockwise are positive and clockwise negative.

If $S = 90°$, the two components in Fig. 6 are perpendicular to the *XY*-plane, and the dihedral angle equals q. Generally, $0 \le S \le 180°$, and the dihedral angle is smaller than q if $S \ne 90°$.

The dihedral angle is determined in the following way using plane trigonometry, vectors and rotation matrices. Define unit vectors along the coordinate axes as

$$\vec{e}_X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_Y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{e}_Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Also, define rotation matrices

$$R_X(\varphi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_X & -\sin\varphi_X \\ 0 & \sin\varphi_X & \cos\varphi_X \end{bmatrix}, \quad R_Y(\varphi_Y) = \begin{bmatrix} \cos\varphi_Y & 0 & \sin\varphi_Y \\ 0 & 1 & 0 \\ -\sin\varphi_Y & 0 & \cos\varphi_Y \end{bmatrix},$$

and

$$R_Z(\varphi_Z) = \begin{bmatrix} \cos\varphi_Z & -\sin\varphi_Z & 0 \\ \sin\varphi_Z & \cos\varphi_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For example, the operation $R_X(\varphi_X)\vec{v}$ rotates the vector $\vec{v} = \begin{bmatrix} v_X & v_Y & v_Z \end{bmatrix}^T$ through an angle φ_X around the *X*-axis to become the vector

$$R_X(\varphi_X)\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_X & -\sin\varphi_X \\ 0 & \sin\varphi_X & \cos\varphi_X \end{bmatrix}\begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix} = \begin{bmatrix} v_X \\ v_Y\cos\varphi_X - v_Z\sin\varphi_X \\ v_Y\sin\varphi_X + v_Z\cos\varphi_X \end{bmatrix}. \tag{1}$$

Now, consider unit vectors that are perpendicular to each of the two components in Fig. 6. A unit vector that is perpendicular to the showing face of the component aligned along the *Y*-axis (labeled *component 1*) results by a counterclockwise rotation around the *Y*-axis of the unit vector along the *Z*-axis, $\vec{e}_Z$, through the angle of the slope, S, of the face

$$\vec{u}_1 = R_Y(S)\vec{e}_Z = \begin{bmatrix} \sin S \\ 0 \\ \cos S \end{bmatrix}. \tag{2}$$

For example, if $S = 90°$, component 1 is perpendicular to the *XY*-plane, and $\vec{u}_1 = \vec{e}_X$.

A unit vector that is perpendicular to the showing face of the other component (labeled *component 2*) results by a counterclockwise rotation of $\vec{u}_1$ through an angle of $180° - q$ around the *Z*-axis:

$$\vec{u}_2 = R_z\left(180° - q\right)\vec{u}_1 = \begin{bmatrix} \cos q & -\sin q & 0 \\ \sin q & -\cos q & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sin S \\ 0 \\ \cos S \end{bmatrix} = \begin{bmatrix} \cos q \sin S \\ \sin q \sin S \\ \cos S \end{bmatrix}. \tag{3}$$

The angle between these unit vectors is $180° - q_{dihe}$, where q_{dihe} is the the dihedral angle, so[3]

$$\cos\left(180° - q_{dihe}\right) = \frac{\vec{u}_1 \cdot \vec{u}_2}{|\vec{u}_1||\vec{u}_2|} = -\cos q \sin^2 S + \cos^2 S. \tag{4}$$

Since $\cos\left(180° - q_{dihe}\right) = -\cos q_{dihe}$, we have

$$\cos q_{dihe} = \cos q \sin^2 S - \cos^2 S. \tag{5}$$

For example, if $S = 90°$ and $q = 90°$, $\cos\left(q_{dihed}\right) = 0$, which implies that $q_{dihed} = 90°$. This may also be confirmed by examining the two components in Fig. 6. When $S = 90°$ and $q = 90°$ the two components are both perpendicular to the XY-plane and to each other, so $q_{dihed} = 90°$. Generally, $0 \le q_{dihed} \le 180°$.

It will be convenient for specifying setup angles for cutting compound miter-joints to have an expression for half the dihedral angle, $q_{dihe}/2$. One expression is $q_{dihe}/2 = (1/2)\arccos\left[\cos q \sin^2 S - \cos^2 S\right]$. An alternative expression is obtained by using the following trigonometric identity for half angles: $\cos\left(f\right) = 2\cos^2\left(f/2\right) - 1$, which implies from Eqn. (5) that

$$2\cos^2\left(q_{dihe}/2\right) - 1 = \left[2\cos^2\left(q/2\right) - 1\right]\sin^2 S - \cos^2 S = 2\cos^2\left(q/2\right)\sin^2 S - 1. \tag{6}$$

Thus,

$$\cos^2\left(q_{dihe}/2\right) = \cos^2\left(q/2\right)\sin^2 S. \tag{7}$$

Since $0 \le q_{dihed}/2 \le 90°$, $0 \le q/2 \le 90°$ and $0 \le S \le 180°$, $0 \le \cos\left(q_{dihed}\right) \le 1$, $0 \le \cos\left(q/2\right) \le 1$ and $0 \le \sin S \le 1$. Consequently,

$$\cos\left(q_{dihe}/2\right) = \cos\left(q/2\right)\sin S, \tag{8}$$

and

$$\frac{1}{2}q_{dihe} = \arccos\left[\cos\left(q/2\right)\sin S\right]. \tag{9}$$

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[3] The 'dot product' and 'cross product' notation is summarized in Section 9.1.

It will also be convenient to have an expression for q_{dihed} in addition to the one in Eqn. (8) for the half angle. One such expression is obtained by squaring both sides of Eqn. (8) and using the half-angle identity $\cos^2(f/2) = (1 + \cos f)/2$ twice to obtain

$$\cos(q_{dihed}) = \cos q \sin^2 S - \cos^2 S. \qquad (10)$$

2.2 Cut planes

Compound-angle joints are commonly formed using either a *miter joint* or a *butt joint*. The appearance of these configurations as seen in the *XY*-plane is shown in Fig. 7.
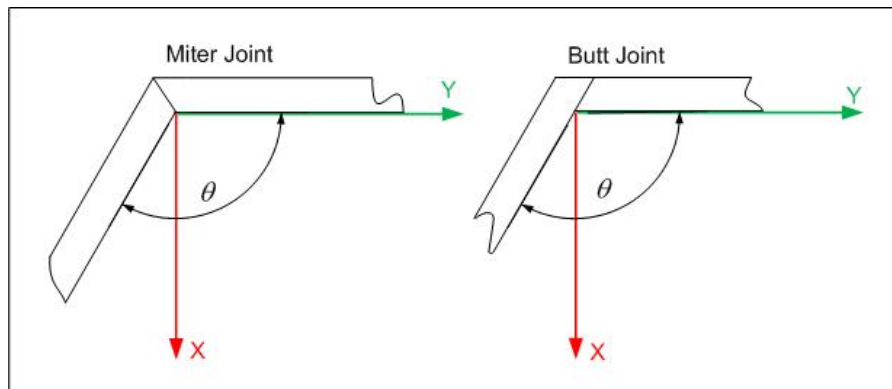

Figure 7. Miter and butt joints viewed in the *XY*-plane

The two components of the compound-angle joint are cut to form these configurations. The complexity when cutting them arises because cutting involves planes that are not parallel to the *XY*-plane. The principal change is in the angle appearing to join the components. It is here that the dihedral angle becomes important.

Shown in Figs. 8 and 9 are the planes that a saw blade[4] must occupy to form the mating parts.

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[4] Here, we think of the saw blade as having a very narrow kerf. Otherwise, the cut plane contains only one side of the blade.
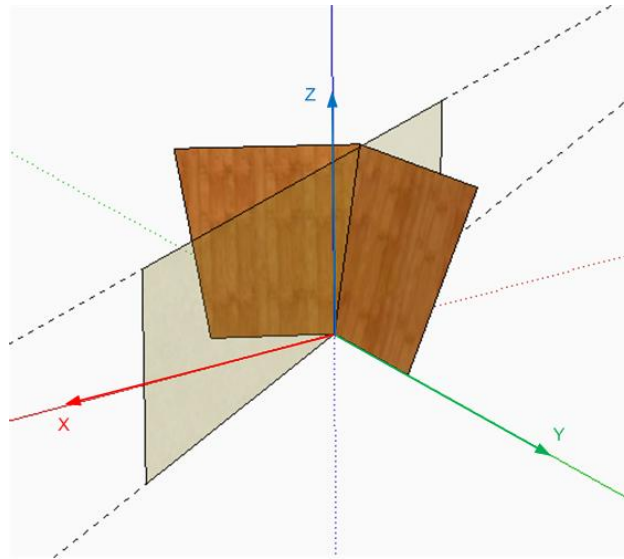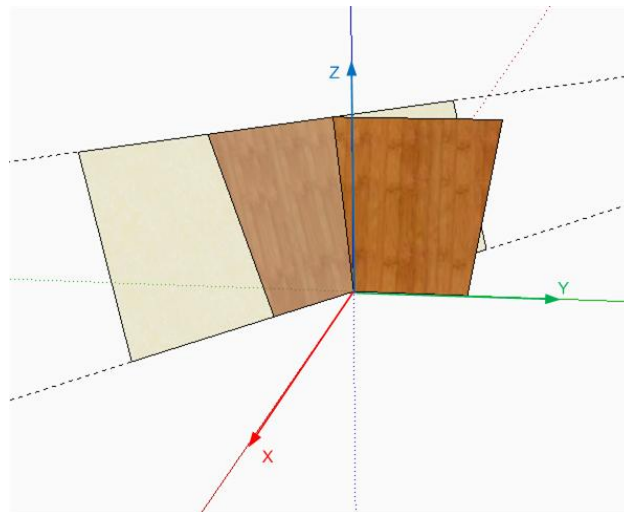
Figure 8. Cutting plane to form a miter joint

Figure 9. Cutting plane for a butt joint

The cutting plane for a miter joint divides in half both the angle q and the dihedral angle q_{dihe} of the compound-angle joint, whereas the cutting plane for a butt joint is oriented so as to contain the show face of component 2 of the compound-angle joint.

3   Development of setup angles using plane trigonometry, vectors and rotation matrices

Here is what we envision needs to happen. We consider that the imaginary cut plane is attached to component 1. The showing face of this component (or of the board or material that is to become component 1) with its associated cut plane needs to be oriented on the surface of the table saw (or other cutting machine) so that the cut plane passes through the plane of the saw blade. This requires that the saw blade be tilted and the miter gauge be adjusted to match the orientation of the cut plane. So, we rotate component 1 and its attached cut plane clockwise about the Y-axis through the angle S. Component then lies on the surface of the table saw, and the rotated unit vector $\dot{u}_1$ will then equal the unit vector $\dot{e}_z$ along the Z-axis. The cut plane will be at an angle to the surface of the table saw of $q_{dihed} / 2$ for a miter joint or angle q_{dihed} for a butt joint. We call the angle that the cut plane makes with the surface of the table-saw surface the *blade angle*, and denote it by $BA°$, measured in degrees. Thus, by combining Eqns. (8) and (10), measured counterclockwise from the surface of the saw table, the blade angle $BA°$ satisfies

$$\cos\left(BA^o\right) = \begin{cases} \cos\left(q/2\right)\sin S, & \text{miter joint} \\ \cos q \sin^2 S - \cos^2 S, & \text{butt joint} \end{cases} \tag{11}$$

The blade angle $BA^o = 90^o$ corresponds to the blade or cut plane being perpendicular to the surface of the saw table. We use the term *blade tilt*, denoted by $BT°$, measured positive counterclockwise, as the angle of the blade or cut plane measured from 90° to the table-saw surface; then, $BT^o = 0^o$ corresponds to a blade or cut plane that is perpendicular to the saw's surface. Thus, $BT^o = BA^o - 90^o$, the negative complement of the blade angle. Since this implies that $\cos\left(BA^o\right) = -\sin\left(BT^o\right)$, the blade tilt-angle satisfies

$$\sin\left(BT^o\right) = \begin{cases} -\cos\left(q/2\right)\sin S, & \text{miter joint} \\ \cos^2 S - \cos q \sin^2 S, & \text{butt joint} \end{cases} \tag{12}$$

It is helpful to be aware of both the blade-tilt angle BT^o and blade angle BA^o because each can be useful. These angles are displayed in Figure 10. The tilt-angle
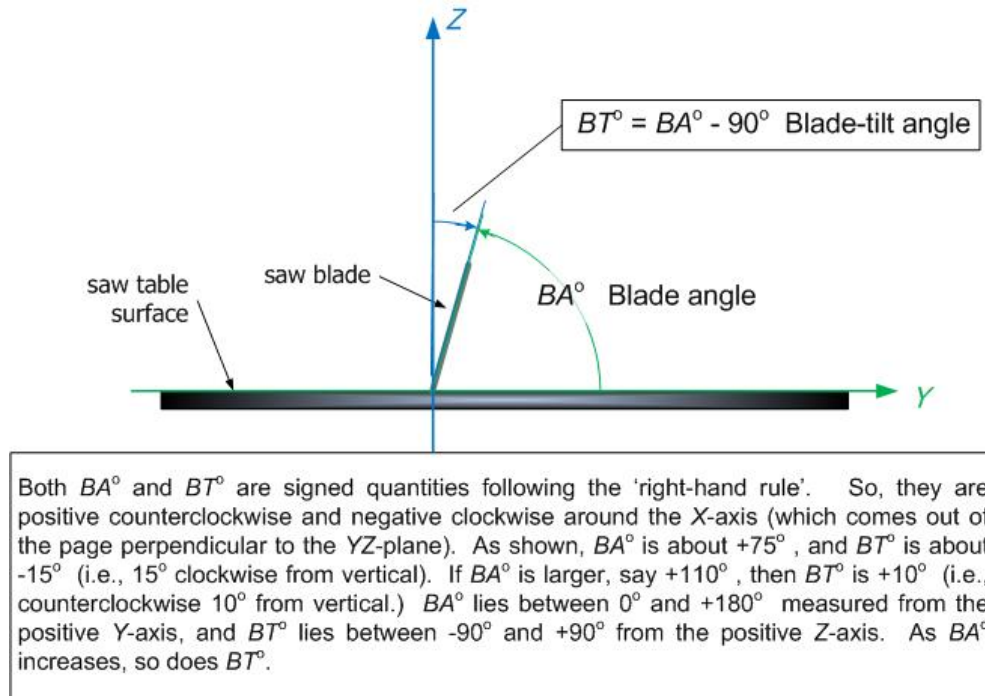
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Figure 10. Blade and blade-tilt angles

is useful because this is the angle displayed on the angle scale that is built into saws made by many manufacturers. However, these scales are coarse, making precise settings difficult. Instead of using the saw's angle scale when more precise settings are needed, it is convenient to set an auxiliary tool, such as a bevel gauge, to the desired blade angle BA^o using an accurate protractor. The bevel gauge is then placed on the table and the blade adjusted to match its angle. An alternative that can be even more precise is to print onto regular printer paper a right triangle with one of the acute angles being BA^o. The printer paper is then glued to a heavy card stock backing, which is cut to form a triangular template that is used to set the blade to angle BA^o.

A rotation of the combined component 1 and its cut plane around the Z-axis is also needed to bring the cut plane into the plane of the tilted saw blade. That angle is the required miter-gauge setting. This we identify in the following way. A unit vector is constructed along the line where the two components meet. This unit vector is rotated clockwise through an angle S around the Y-axis along with component 1 and its attached cut plane. The resulting vector is then rotated around the Z-axis through an angle that yields a vector with an X component of zero. The angle required to accomplish yields the miter-gauge angle.

The vector that results by forming the cross product between $\vec{u}_1$ and $\vec{u}_2$,

$$\vec{u}_1 \times \vec{u}_2 = \begin{bmatrix} -\sin q \sin S \cos S \\ \cos q \sin S \cos S - \sin S \cos S \\ \sin q \sin^2 S \end{bmatrix} = \sin(q_{dihed})\vec{n}, \qquad (13)$$

lies in the cutting plane (for both miter and butt joints) along the mating line of components 1 and 2; $\vec{n}$ is a unit vector that lies along the mating line. Clockwise rotation of this vector about the Y-axis through angle S yields

$$\vec{w} = R_Y(-S)\left[\vec{u}_1 \times \vec{u}_2\right]$$

$$= \begin{bmatrix} \cos S & 0 & -\sin S \\ 0 & 1 & 0 \\ \sin S & 0 & \cos S \end{bmatrix} \begin{bmatrix} -\sin q \sin S \cos S \\ \cos q \sin S \cos S - \sin S \cos S \\ \sin q \sin^2 S \end{bmatrix}$$

$$= \begin{bmatrix} -\sin q \sin S \cos^2 S - \sin q \sin^3 S \\ -\cos q \sin S \cos S - \sin S \cos S \\ \sin q \sin^2 S \cos S + \sin q \sin^2 S \cos S \end{bmatrix} \qquad (14)$$

$$= \begin{bmatrix} -\sin q \sin S \\ -\cos q \sin S \cos S - \sin S \cos S \\ \sin q \sin^2 S \cos S + \sin q \sin^2 S \cos S \end{bmatrix}.$$

The vector $\vec{w}$ is in the XY-plane aligned along the mating line of the two components, as shown in Figure 11 for component 1.
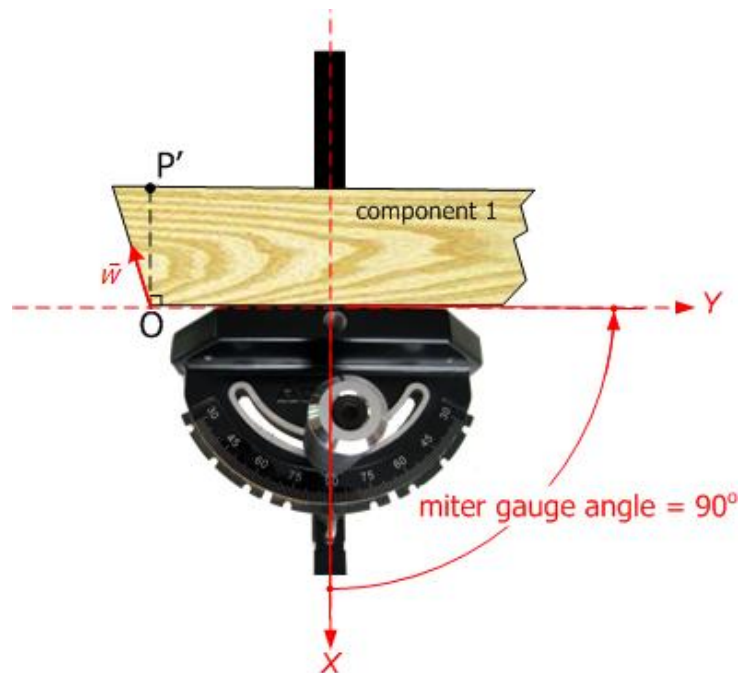

Figure 11. Component 1 positioned flat on the XY-plane

We now seek the angle f_z such that $\vec{e}_Y \cdot R_z(f_z)\vec{w} = 0$. For this angle, the vector $R_z(f_z)\vec{w}$, and therefore the cut plane, is perpendicular to the Y-axis, as shown in Figure 12.
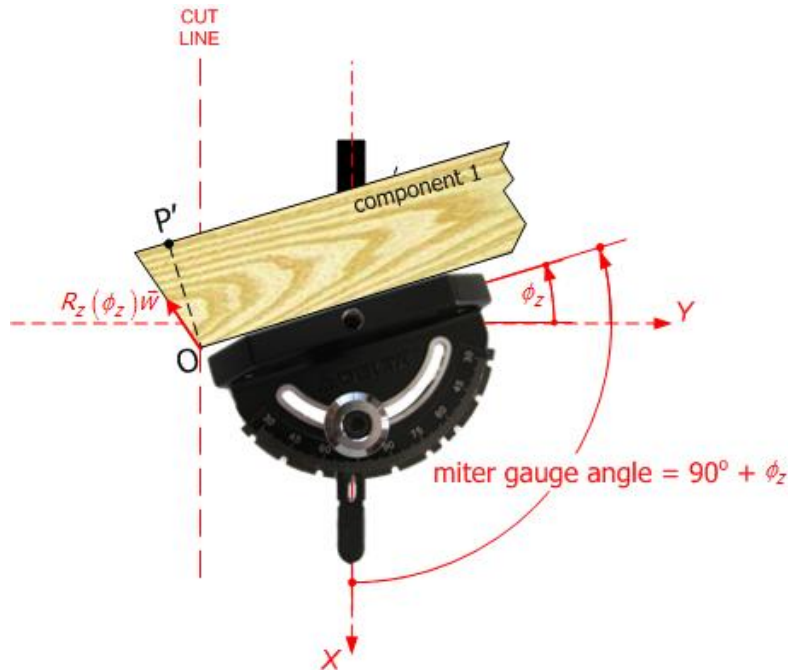
Figure 12. Component 1 rotated

The rotation around the Z-axis yields

$$R_z(f_z)\vec{w} = \begin{bmatrix} \cos j_z & -\sin j_z & 0 \\ \sin j_z & \cos j_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin q \sin S \\ -\cos q \sin S \cos S - \sin S \cos S \\ \sin q \sin^2 S \cos S + \sin q \sin^2 S \cos S \end{bmatrix}. \qquad (15)$$

Setting the Y component of the resulting vector equal to zero yields

$$\sin f_z \sin q \sin S = -\cos f_z (\cos q \sin S \cos S + \sin S \cos S).$$

Thus,

$$\tan f_z = \frac{\sin f_z}{\cos f_z} = -\frac{(1+\cos q)\cos S}{\sin q}. \qquad (16)$$

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We now use the half-angle formulas $\sin q = 2\sin(q/2)\cos(q/2)$ and $1 + \cos q = 2\cos^2(q/2)$ to obtain

$$\tan f_z = -\frac{\cos S}{\tan(q/2)}. \tag{17}$$

The miter-gauge angle[5], shown as MG^o in Fig. 12, is given in terms of f_z by $MG^o = 90^o + f_z$, which can be less or greater than 90° depending on the sign of f_z. Thus, from Eqn. (17)

$$\tan(MG^o) = \tan(90^o + f_z) = -\frac{1}{\tan f_z} = \frac{\tan(q/2)}{\cos S}. \tag{18}$$

Care is needed with the inverse tangent-function when determining the miter-gauge angle with this expression. The version of the inverse tangent-function to be used is:

$$MG^o = \arctan 2\big(\tan(q/2), \cos S\big), \tag{19}$$

where

$$\arctan 2(y, x) = \begin{cases} \arctan(y/x), & x > 0 \\ \arctan(y/x) + 180^o, & y \geq 0, x < 0 \\ \arctan(y/x) - 180^o, & y < 0, x < 0 \\ 90^o, & y > 0, x = 0 \\ -90^o, & y < 0, x = 0 \\ \text{undefined}, & y = 0, x = 0 \end{cases} \tag{20}$$

Expressions for the blade and miter setup angles for compound-angle joints are developed in this section using plane trigonometry, vectors and rotation matrices. Eqn. (12) is the expression for the blade-tilt angle, and Eqn. (18) is for the miter-gauge angle. In each of these expressions, the angle q is the angle in the XY-plane between the two components forming the compound-angle joint, as shown in

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[5] Note here that the miter-gauge angle MG^o is measured as an angle about the Z-axis, with 90° corresponding to the miter gauge set perpendicular to the plane of the cutting blade. The scale on miter-gauge tools are often marked with 90° corresponding to perpendicular to the cutting blade and, further, the scale is marked symmetrically either side of 90°. Care is therefore needed when converting MG^o into scale readings.

Figure 6. Expressions for the setup angles are developed in another way in the next section.

4   Development of setup angles using spherical trigonometry

Bill Gottesman formulated the ideas for this alternative development by using spherical trigonometry. Bill routinely uses spherical trigonometry for his designs of novel sundials, and the motivation for pursuing this development originated from a conversation between the authors at the Annual Conference of the North American Sundial Society held in Indianapolis, IN, in August 2014. We offer both the development of the previous section based on plane trigonometry and the development of this section using spherical trigonometry because each conceptualization provides its own distinct insights that may be helpful to others interested in compound-angle joinery.

Spherical trigonometry plays an important role in many applications, including navigation, astronomy, and surveying. We will indicate its use in compound-angle joinery. First some terminology. A circle is formed at the intersection of a plane with the surface of a sphere. Such a circle is called a *great circle* when the intersecting plane passes through the center of the sphere; otherwise it is called a *small circle*. For example, the equator is a great circle on the earth's surface (when the earth is approximated as a sphere), dividing the earth into its northern and southern hemispheres. Great circles partition the earth into zones of longitude, and small circles partition it into zones of latitude. Spherical trigonometry deals with polygonal shapes that occur on the surface of a sphere when multiple great circles intersect. As seen in Fig. 10, three great circles can intersect to form a spherical
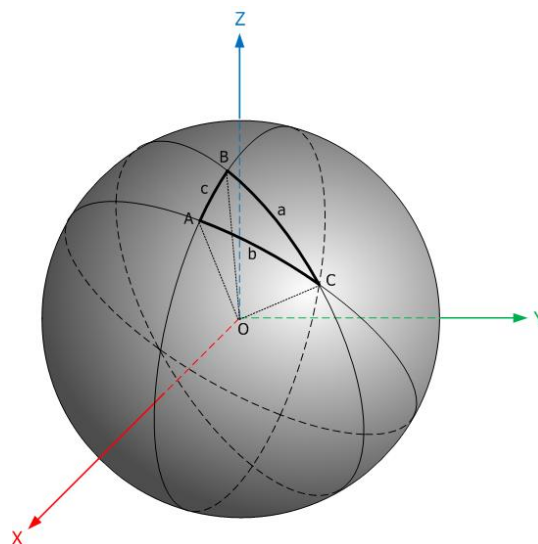

Figure 13. Spherical triangle

triangle. Four can form a spherical square, five a spherical pentagon, etc. Spherical pentagons and hexagons are the shapes seen on the surface of the soccer ball in Fig. 11. Three planes intersecting to form a spherical triangle can help to explain table-saw setup angles when forming objects requiring compound angle joinery.

Figure 14. Soccer ball

Fig. 10 shows three great circles that intersect to form a spherical triangle with vertices labeled *A*, *B* and *C*. The angle *C-A-B* at vertex *A* is also referred to as the angle *A*, which is measured in angular units of degrees or radians; likewise for the angles at vertices *B* and *C*. The three sides of the spherical triangle are also measured in angular units. The side opposite of the vertex *A* is labeled *a*. The size of *a* is that of the angle *B-O-C*. Similarly, the sizes of the sides labeled *b* and *c* are those of the angles *A-O-C* and *A-O-B*, respectively. A plane that is tangent to the sphere at the point of vertex *A* is perpendicular to the radial line *O-A*. Therefore, any line in that plane which passes through the point of tangency is perpendicular to that radial line. In particular, consider two lines in the tangent plane, one that is also in the plane defined by *A-O-B* and the other in the plane defined by *A-O-C*. The angle between these lines is the dihedral angle of the two intersecting planes containing *A-O-B* and *A-O-C*. Thus, the angle *A* associated with vertex *A* is the dihedral angle of the two planes that intersect along the radial line *O-A*.

Spherical trigonometry deals with relationships between the six angles *A*, *B*, *C*, *a*, *b* and *c*. Here, we summarize these relationships found in standard textbooks [8, 9, 11] and many websites [7, 10]. The fundamental equations, called the *law of cosines*, are:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \qquad (21)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Six angles are associated with any spherical triangle. These fundamental equations permit the determination of all six by knowing any three of them. Manipulation of these fundamental equations yields:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \qquad (22)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Daniel Wenger gives a derivation of the law of cosines by using the rotation matrices of Section 2.1. Many other relationships can be derived from the law of cosines by using trigonometric identities [8-11].

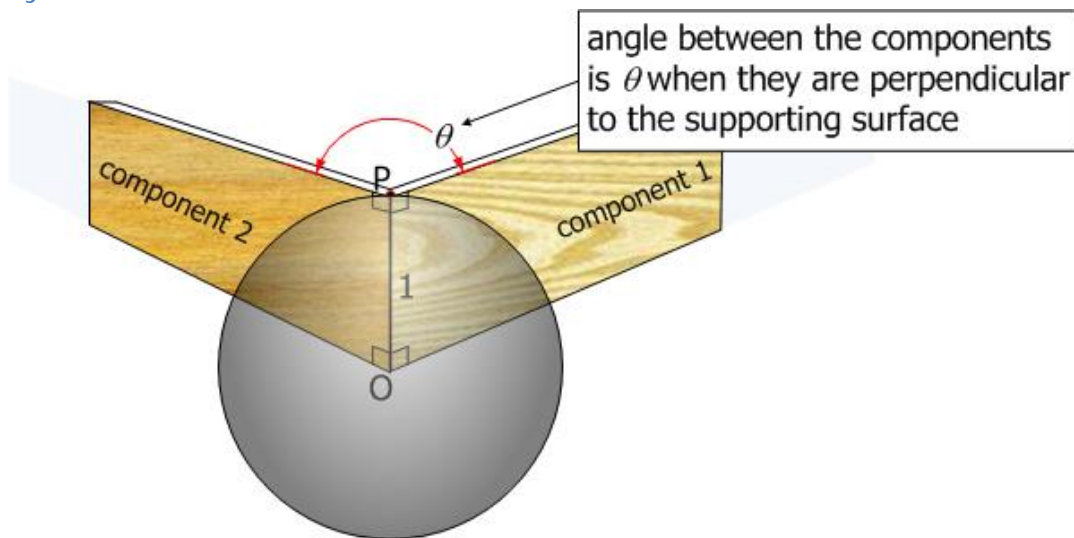## 4.1 Placing a compound-angle joint in the geometry of spherical trigonometry

Figure 15. Initial geometry

Shown in Figure 15 are portions of planes containing two components that will form a compound-angle joint once they are tilted through an angle $S°$. They are pres-

ently oriented 90° to the *XY*-plane containing the equatorial great circle and making an angle *q* to one another. The center of the sphere is located at the point labeled *O* in the *XY*-plane where the two component planes intersect. A point labeled *P* lies 1 unit from *O* along the *Z*-axis. Now suppose the two components Figure 15 are tilted by an angle *S* measured from the *XY*-plane, as shown in Figure 16. The point *P* is split into two reference points, labeled *P'* and *P''*, with each moving in place on its respective component through 90° - S^o degrees as the components are tilted through S^o degrees. The tilted components are then extended until they intersect and form a dihedral angle equal to q_{dihed} .
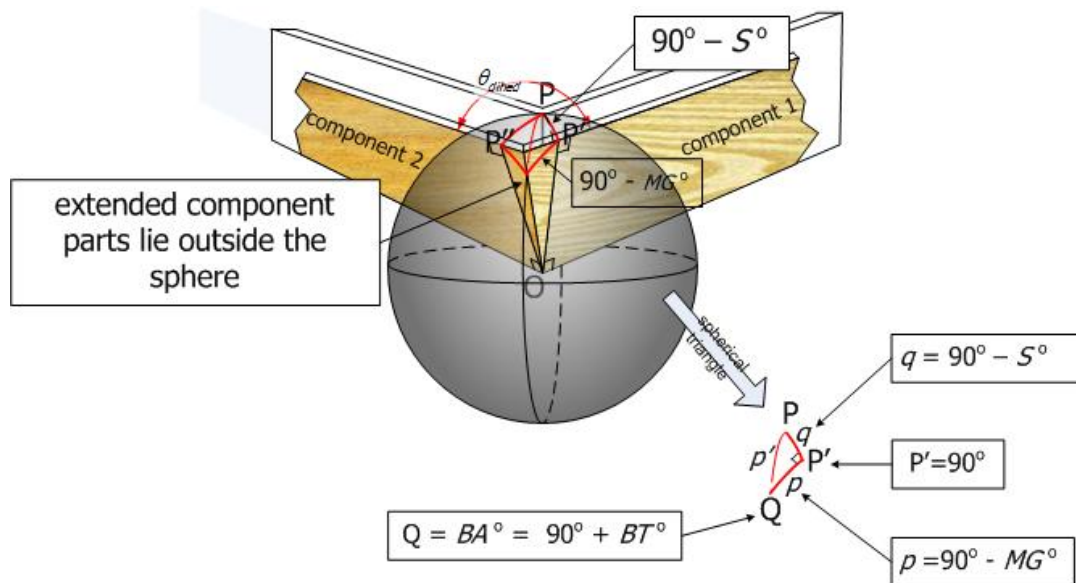

Figure 16. Components tilted

Figure 17 shows component 1 resting on the surface of a table saw and resting against a miter gauge. The gauge on the left is set at $MG^o = 90^o$, and the one on the right is set at an angle MG^o so that the line where the two components are to be joined is aligned with the cutting line of the saw blade.
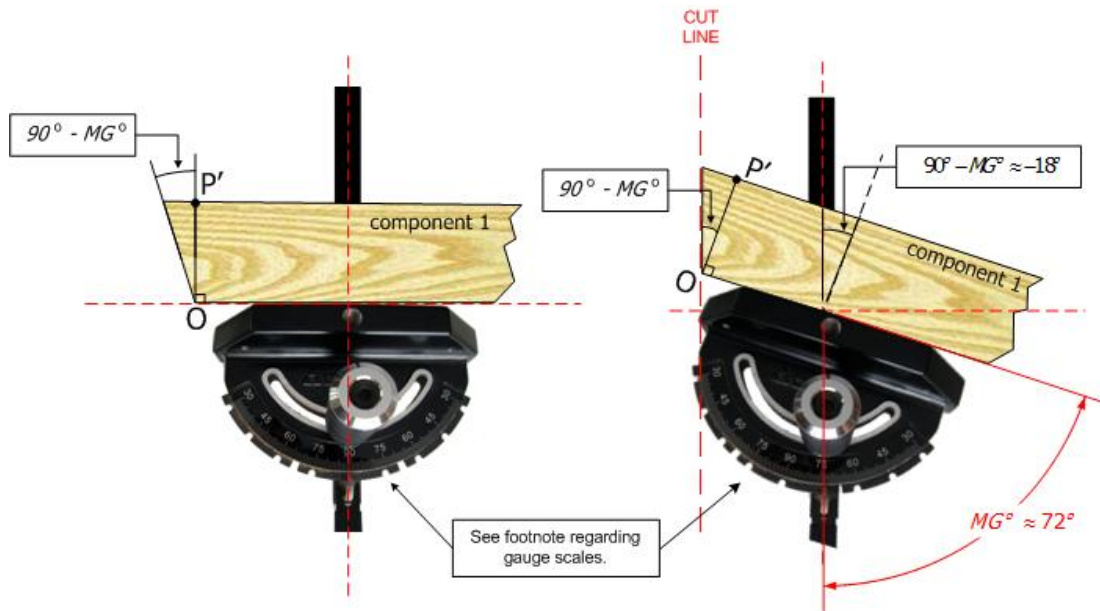
Figure 17. Component 1 resting on the surface of a table saw against a miter gauge

Important arcs that are segments of great circles are shown in Figure 16. These form a spherical triangle with vertices P-Q-P'.   The arc angle p is the complement of the miter-gauge angle MG^o for making compound-angle joints, and the vertex angle at Q is the blade angle BA^o for the compound miter- and butt-joints. These angles can be determined from the spherical triangle P-Q-P' by using the law of cosines.

In the spherical triangle P-Q-P', angles at the vertices are $P = \left(180^o - q\right)/2$, $Q = q_{dihed}/2$, and $P' = 90^o$.   Also, the arc P-P', which we label q, equals $90^o - S^o$. From Eqn. (20) for spherical triangles, the angle Q satisfies

$$\cos Q = -\cos P \cos P' + \sin P \sin P' \cos q .$$

which becomes

$$\cos\left(q_{dihed}/2\right) = -\cos\left(180^o - q\right)\cos\left(90^o\right) + \sin\left(90^o - q/2\right)\sin\left(90^o\right)\cos\left(90^o - S\right)$$
$$= \cos\left(q/2\right)\sin S . \tag{23}$$

This is the same result as in Eqn. (8). It then follows that the blade angle BA^o for compound-angle miter- and butt-joints is given by

$$\cos\left(BA^o\right) = \begin{cases} \cos\left(q/2\right)\sin S, & \text{miter joint} \\ \cos q \sin^2 S - \cos^2 S, & \text{butt joint} \end{cases} \tag{24}$$

This is the same as Eqn. (11). To determine the arc P'-Q, which we label p, we again use the cosine law, Eqn. (21), to obtain

$$\cos P = -\cos P'\cos Q + \sin P'\sin Q \cos p .$$

Thus,

$$\cos\left(90^0 - q/2\right) = -\cos\left(90^0\right)\cos\left(q_{dihed}/2\right) + \sin\left(90^0\right)\sin\left(q_{dihed}/2\right)\cos p$$
$$= \sin\left(q_{dihed}/2\right)\cos p. \tag{25}$$

The miter-gauge angle for a compound-angle joint is equal to the complement of the arc-angle p, so

$$\cos\left(90^o - MG^o\right) = \sin\left(MG^o\right) = \frac{\sin\left(q/2\right)}{\sin\left(q_{dihed}/2\right)} . \tag{26}$$

This is an alternative expression for the miter-gauge angle from that in Eqn. (18). To see that the two expressions yield identical results requires some manipulations using trigonometric identities. Start by squaring Eqn. (26) and using Eqn. (23) to get

$$\cos^2\left(MG^o\right) = 1 - \sin^2\left(MG^o\right) = 1 - \frac{\sin^2\left(q/2\right)}{1 - \cos^2\left(q/2\right)\sin^2 S} = \frac{\cos^2\left(q/2\right)\cos^2 S}{1 - \cos^2\left(q/2\right)\sin^2 S} .$$

Then,

$$\frac{\sin^2\left(MG^o\right)}{\cos^2\left(MG^o\right)} = \tan^2\left(MG^o\right) = \frac{\sin^2\left(q/2\right)}{\cos^2\left(q/2\right)\cos^2 S} = \frac{\tan^2\left(q/2\right)}{\cos^2 S} .$$

The square root then shows that Eqns. (26) and (18) are equivalent expressions for the miter-gauge angle for forming a compound-angle joint.

5   Summary of Results

Assume that the two components of a compound-angle joint meet at an angle q in the XY-plane. A special case arises when the components are part of a box whose shape in the XY-plane is a regular polygon (eg. square, hexagon) having N sides;

then $q = 180^0 (N - 2)/N$. The miter-gauge angle MG^0 is referenced to be 90^0 when it is perpendicular to the plane of the cutting blade, and the blade-tilt angle BT^0 is referenced from the XY-plane (surface of the saw table). Assume that the component parts slope S^0 from the XY-plane. Then, the blade-tilt angle satisfies

$$\sin(BT^o) = \begin{cases} -\cos(q/2)\sin S, & \text{miter joint} \\ \cos^2 S - \cos q \sin^2 S, & \text{butt joint.} \end{cases} \tag{27}$$

The dihedral angle q_{dihed} of the compound-angle joint is equal to $2*BA^0$ for a miter joint and BA^0 for a butt joint. The miter-gauge angle satisfies

$$\tan MG^o = \frac{\tan(q/2)}{\cos S}, \tag{28}$$

which is Eqn. (18). Alternatively, and equivalently, the miter-gauge angle also satisfies

$$\sin(MG^o) = \frac{\sin(q/2)}{\sin(q_{dihed}/2)}, \tag{29}$$

which is Eqn. (26). For regular polygonal boxes having N sides, $q = 180^o (N - 2)/N$, these expressions become

$$\sin(BT^o) = \begin{cases} -\cos\left[90^o (N - 2)/N\right]\sin S, & \text{miter joint} \\ \cos^2 S - \cos\left[90^o (N - 2)/N\right]\sin^2 S, & \text{butt joint} \end{cases} \tag{30}$$

and

$$\tan MG^o = \frac{\tan\left[90^o (N - 2)/N\right]}{\cos S} \tag{31}$$

or, alternatively

$$\sin(MG^o) = \frac{\sin\left[90^o (N - 2)/N\right]}{\sin(q_{dihed}/2)}, \tag{32}$$

where $q_{dihed}/2 = BA^o$ for a miter joint in Eqn. (29).

6   Computer Implementations

6.1   Internet Implementations

A number of Internet sites provide interactive calculators that produce values for saw setup angles for compound-angle joinery. References [3] and [4] are examples.

6.2   Frink Implementation

FRINK is a scientific programming language that is useful for creating applications that run on many devices such as smart-phones (including Android, iPhone), Windows, MacOS, Linux, and others. It was created and is maintained by Alan Eliasen and made available for free on Google's app website (Google Play) and via Eliasen's website at http://futureboy.us/frinkdocs/, which also has user documentation and many example applications. The Appendix has a listing of a FRINK procedure for calculating and displaying table-saw setup-angles for compound-angle joints for boxes. A screen shot of the displayed results when run on an Android smart-phone for a box with $N = 7$ sides and a slope of $S = 83°$, which are the parameters for the heptagon vase displayed in Figure 2, is in Figure 18.

```
CompoundAngleJoints_Ver2.frink

SETUP ANGLES FOR BOXES WITH COMPOUND-ANGLE JOINTS
Input: User Values
 Number of Sides = 7
 Slope of Sides (from horizontal) = 83°
Output: Saw Setup Values
 Miter Gauge Angle (from plane of blade) = 86.64 °
   Complementary Miter-Gauge Angle (|90°-|MG||]) = 3.36 °
   Supplementary Miter-Gauge Angle (|180°-|MG||) = 93.36 °
 Blade Tilt Angle for Miter Joint (from perpendicular to blade surface) BT = -25.51 °
 Blade Angle for Miter Joint (from saw surface to blade surface) (|90°-|BT||) = 64.49
°
   Supplementary Blade Angle for Miter Joint (|180°-|BT||) = 115.51 °
 Blade Tilt Angle for Butt Joint (from perpendicular to saw surface) BT = 38.98 °
 Blade Angle for Butt Joint (from saw surface to blade surface) (|90°-|BT||) = 128.98
°
   Supplementary Blade Angle for Butt Joint (180°-|BT|) = 51.02 °
Output: Dihedral Angle (smallest angle between sides) = 128.98 °

                              Close
```

Figure 18.   Screen shot from an Android smart-phone showing the setup angles for making the heptagon vase of Fig. 2

7   Examples

Table 1. Examples (side slope S is measured from the saw table surface, blade tilt BT is measured from the perpendicular to the saw table surface, and miter-gauge angle MG is measured from the plane of the saw blade)

number of sides N	component angle $q^0 = 180^0 (N - 2) / N$	slope of sides, S^0	miter-gauge angle MG^0	miter joint saw-blade tilt BT^0	butt joint saw-blade tilt BT^0
4	90°	90°	90°	-45°	0
4	90°	98°	97.9°	-44.5°	1.1°
6	120°	30°	63.4°	-14.5°	61.0°
7	128.6°	83°	86.6°	-25.5°	39°

8   References

8.1   References related to compound-angle joinery

1. D. L. Snyder, *Cutting Compound Miters on a Table Saw*,
    http://dls-website.com/documents/WoodworkingNotes/Compound%20Miters.pdf

2. Greg Kimnach, *Woodworking – compound miter angle tutorial*. This contains a short explanation of saw setup angles. It is available as an Internet document at
    http://kimnach.org/woodworking/Compound%20miters/compoundangle.htm.

3. SBE Builders, *Development of rake Crown Molding Miter Angles Using Geometry*. This Internet site contains a discussion of compound-angle joinery for crown molding.   It is available at http://www.sbebuilders.com/crown/.

4. Janesson, Compound Saw Calculator. This Internet site contains a discussion of compound-angle joinery. Interactive calculators are provided to display saw setup angles. It is available at http://jansson.us/jcompound.html. The HTML code for this website exhibits the formulas used for setup angles and provides several helpful comments.

5. Chris Glad, *Compound Angle Calculators*. Interactive calculators are provided to display setup angles. They are available on the Internet at http://www.pdxtex.com/canoe/compound.htm.

6. *Compound Miter Template Generator*. An interactive calculator is provided to display setup angles and print templates. It is available on the Internet at http://www.blocklayer.com/compoundmitereng.aspx.

8.2   References related to spherical trigonometry

7. Daniel L. Wenger, *Derivation of the Spherical Law of Cosines and Sines using Rotation Matrices*. Available on the Internet at http://www.wengersundial.com/new/resources/Mathematics/SpherTrig.pdf.

8. Daniel A. Murray, *Spherical Trigonometry*, Longmans, Green and Co., 1908. This is available as an Internet Archive Organization eBook at https://archive.org/details/sphericaltrigono00murrrich.

9. I. Todhunter, *Spherical Trigonometry: For the Use of Colleges and Schools, With Numerous Examples*, McMillan and Co., 1886. This is available as a Project Gutenberg eBook at http://www.gutenberg.org/files/19770/19770-pdf.pdf.

10. *Spherical Trigonometry*, an Internet Wikipedia resource http://en.wikipedia.org/wiki/Spherical_trigonometry

11. A. Albert Klaf, *Trigonometry Refresher*, Dover Publications Inc., 2005. This contains a chapter on spherical trigonometry.

9   Appendices

9.1   Notation

Two forms of multiplication for vector-valued quantities are used in development. These are defined as follows. Let $\vec{a}$ and $\vec{b}$ be three-dimensional vectors,

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The *dot product*, denoted by $\vec{a} \cdot \vec{b}$, is a number defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

The *cross product*, denoted by $\vec{a} \times \vec{b}$, is a vector defined by

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

Discussion of these products and some of their properties is given at the websites:

http://en.wikipedia.org/wiki/Dot_product
http://en.wikipedia.org/wiki/Cross_product

A property of the dot product that we use is:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos q_{ab}$$

where $|\vec{a}| = \sqrt{a_X^2 + a_Y^2 + a_Z^2}$ and q_{ab} is the angle between the vectors $\vec{a}$ and $\vec{b}$. Also, for the cross product,

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin q_{ab} \vec{n}$$

where $\vec{n}$ is a unit vector that is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ and oriented in a direction that is consistent with the right-hand rule.

9.2 Frink procedure for calculating setup angles

---------- start FRINK procedure -----------

```
//  Calculates table saw settings for
//cutting sides of an multi-sided vessel
//having sides that slope.
//    N    number of sides
//    S    angle of the sides from horizontal (degrees)
//    MG   miter gauge angle (degrees) from plane of blade
//    BT   blade tilt angle (degrees) from saw surface
//Created By: D. L. Snyder  26 Feb 2011
//Revised: 27 Oct 2013 (added output of dihedral angle)
//Revised: 29 Aug 2014 (added butt joints, corrected Corner_Angle)
//Revised: 1 Dec 2015 (corrected coordinates for miter guage angle)
//
//start………
```

```
//user inputs
  [N, S] = input["Vessel or Box Information",
          ["Number of sides",
           "Angle of sides from horizontal (degrees)"]]
//evaluate constants
    S_deg = eval[S]
    N_num = eval[N]
    Corner_Angle = 180*(N_num-2)/N_num
    a1 = sin[S_deg degrees]
    a2 = cos[Corner_Angle/2 degrees]
    b1 = cos[S_deg degrees]
    b2 = tan[Corner_Angle/2 degrees]
//evaluate outputs
    BT_MiterJoint = arcsin[-a1*a2]
    BA_MiterJoint = pi/2-abs[BT_MiterJoint]  //complement of BT_MiterJoint
             BA_MJsupplement = pi - abs[BA_MiterJoint]
  //
    BA_ButtJoint = 2*BA_MiterJoint
           BA_BJsupplement = pi - abs[BA_ButtJoint]
        BT_ButtJoint = abs[pi/2 - abs[BA_ButtJoint]] //complement of
BA_ButtJoint
  //
    MG = arctan[b2,b1]  //angle from plane of blade
           MGcomplement = abs[pi/2 - abs[MG]]
        MGsupplement = abs[pi - abs[MG]]
  //
    DihedralAngle = BA_ButtJoint
// display results
    println[" SETUP ANGLES FOR BOXES WITH COMPOUND-ANGLE JOINTS"]
    println[" Input: User Values"]
     println["  Number of Sides = " + N_num]
      println["  Slope of Sides (from horizontal) = " + eval[S] + "\u00B0"]
// '\u00B0' is the unicode for the degree symbol
    println[" Output: Saw Setup Values"]
```

```
    println["  Miter Gauge Angle (from plane of blade) = " + format[MG,"\u00B0",
2]]

       println["    Complementary Miter-Gauge Angle (|90\u00B0-|MG||) = " +
format[MGcomplement,"\u00B0", 2]]

       println["    Supplementary Miter-Gauge Angle (|180\u00B0-|MG||) = " +
format[MGsupplement,"\u00B0", 2]]

    println["  Blade Tilt Angle for Miter Joint (from perpendicular to blade surface)
BT = " + format[BT_MiterJoint,"\u00B0", 2]]

    println["  Blade Angle for Miter Joint (from saw surface to blade surface)
(|90\u00B0-|BT||) = " + format[BA_MiterJoint,"\u00B0", 2]]

       println["   Supplementary Blade Angle for Miter Joint (|180\u00B0-|BT||) = "
+ format[BA_MJsupplement,"\u00B0", 2]]

         println["  Blade Tilt Angle for Butt Joint (from perpendicular to saw surface)
BT = " + format[BT_ButtJoint,"\u00B0", 2]]

    println["  Blade Angle for Butt Joint (from saw surface to blade surface)
(|90\u00B0-|BT||) = " + format[BA_ButtJoint,"\u00B0", 2]]

       println["   Supplementary Blade Angle for Butt Joint (180\u00B0-|BT|) = " +
format[BA_BJsupplement,"\u00B0", 2]]

        println[" Output: Dihedral Angle (smallest angle between sides) = " +
format[DihedralAngle,"\u00B0", 2]]


//.........end
```

---------- end FRINK procedure -----------