Calculus Meets String Inlay

Donald L. Snyder

As far as I have been able to discover, the application of calculus in woodworking is very rare. Even so, I found its use helpful for my project involving a decorative border made with string (Def. 2 of *string*) inlay, and it is that which I will describe. I will say right off that calculus is unnecessary for completing the project because there is a simpler alternative (based on Def. 1 of *string*), but calculus makes for an entertaining exercise in mathematics used in woodworking, entertaining at least to me.

The project is making a small table with a top having decorative-line inlay. Here is a picture of my completed table top.


Some may recognize the inlay pattern. It is a design by the prominent teacher of woodworking, Steve Latta, and here is a picture of him holding his version:

In Ref. [1], Latta describes in detail how to make the lotus flowers appearing in both of our projects by using a small router to follow a pattern made of ½-inch thick MDF [1].

I used Latta's pattern-following method to inlay the undulating border of my table top. This involves making the pattern for the wavy line, then using the pattern and a small router to excavate the wavy channels to hold the string inlay, and then insetting the string in those channels. For the string, I used strips of English-sycamore veneer about 1/32 inch thick, about 1/8 inch wide and several inches long. The channels were made using Dremel router with a 1/32-inch diameter end mill set to a depth of slightly less than 1/8 inch.

The wavy lines of the border pattern are cosinusoidal functions having a peak-to-peak amplitude of 0.25 inch and a period of 1.5 inch. Thus, the equation of the function is $0.125\cos(2\pi x/1.5)$. The peaks and valleys of the function are separated by one-half period, which is 0.75 inches. Two cosinusoidal functions differing in phase by $180°$ comprise the border.
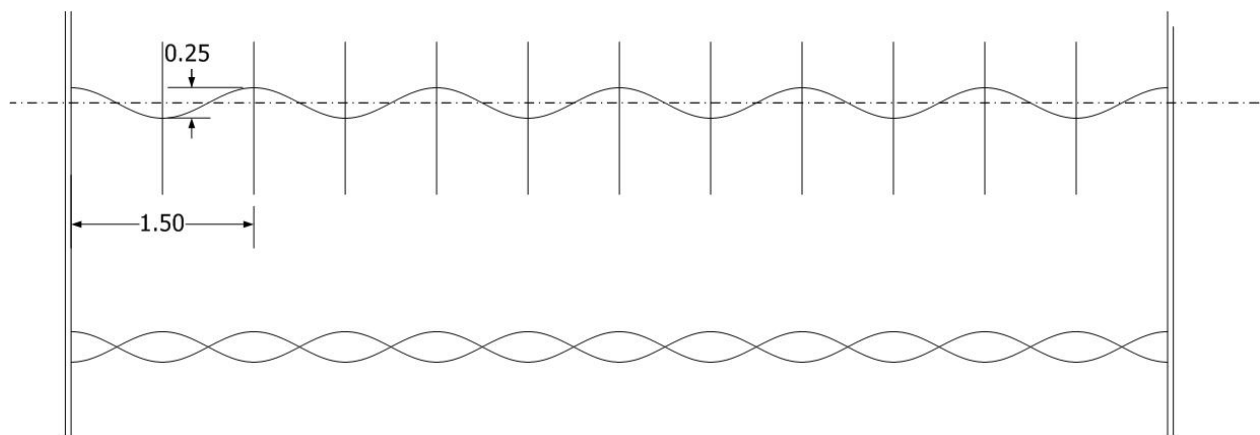
I had a reusable partial-pattern made of ¼-inch thick plastic; this was made by Derek Diepenbrock of ABS Engraving with a CNC laser cutter [2]. This partial pattern was then used to make a full-sized pattern of ½-inch thick MDF. These patterns are shown below.
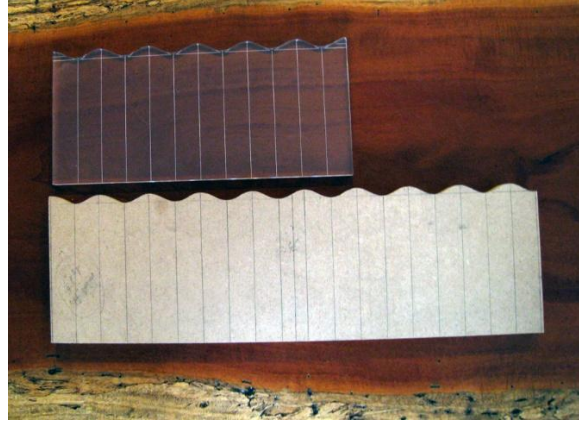
Latta's pattern-following method for string inlay was used with the full-sized MDF pattern to make the borders of my table top. The channels of the two cosinusoidal function are cut one at a time. The first channel is cut, and the stringing for it is inserted, glued in place and then leveled to the surface of the table top. Then the second channel is cut, and the second piece of stringing inserted, glued and leveled.

One issue that arises in making the cosinusoidal inlay with strips cut from thin wood-veneer is that the wood is brittle, tending to splinter or break as it is bent to follow the shape of the undulating function. As Latta describes in Ref. [1], this can be circumvented by heating the veneer strips at the locations of high curvature and shaping the strip to follow that of the channel where it is to be placed. This requires heating and shaping the linear strip of veneer successively, one after another, at the peaks and valleys of the cosinusoidal function. My approach for doing this was to place pencil marks on the linear strip of veneer at the locations of the extreme peaks and valleys; that is, where the maxima and minima occur. These extreme points are to be 0.75 inches apart once the strip is bent into the cosinusoidal shape, but how far apart are they before the strip is bent? Answering this question is where a bit of calculus comes in handy. What is needed is the path length from an extreme peak to the adjacent extreme valley for a function defined by $y(x) = 0.125\cos(2\pi x / 1.5)$. From elementary calculus, the length of a path, call it $L(x_1, x_2)$, along the graph of a function $y(x)$ between two abscissa values, x_1 and x_2 is given by [3]

$$L(x_1, x_2) = \int_{x_1}^{x_2} \sqrt{1 + (dy(x)/dx)^2}\, dx. \tag{1}$$

Thus, the path length between the peak at $x = 0$ and the next adjacent valley, at $x = 0.75$, of the cosinusoidal border is given by

$$L(0,0.75) = \int_0^{0.75} \sqrt{1 + \left(-(\pi/6)\sin(2\pi x/1.5)\right)^2}\, dx, \tag{2}$$

where the derivative formula $d(a\cos(bx))/dx = -ab\sin(bx)$ was used. This is the length along the dark blue segment of the graph of $y(x)$ in the following figure[1].
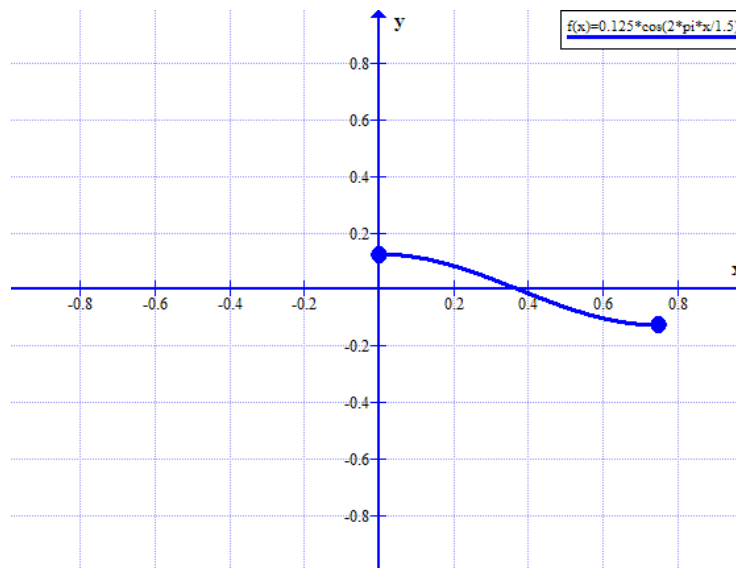
The integral in Eq. (2) is not easily evaluated directly. Instead I used a finite-difference approximation and evaluated that numerically using Microsoft Excel. The approximation is

$$L(0,0.75) \approx \sum_{n=0}^{\lceil 0.75/\Delta x\rceil} \sqrt{1 + \left[(\pi/6)\sin(2\pi n\Delta x/1.5)\right]^2}\, \Delta x. \tag{3}$$

The error with this approximation decreases as Δx decreases. For $\Delta x = 0.01$, the approximation yields $L \approx 0.809$ inches, and for $\Delta x = 0.001$, $L \approx 0.800$. The approximate value using the more sophisticated approximation to the integral in (2) that is implemented in [4] yields $L \approx 0.799$. The software *Graph* also evaluates the path length as 0.799. For the purpose of marking the locations for the heating and bending a piece of inlay stringing, use of $L \approx 0.8$ is more than adequate provided the length of the stringing is not so long that the

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[1] The graph was produced using the software *Graph* written by Ivan Johansen and available from http://www.padowan.dk/.

errors accumulate to a significant amount. I used the 0.8 approximation for the table top. A divider set to this half-cycle distance was used with a pencil to mark the extreme points (the maxima and minima of the cosinusoidal function) on a long, straight piece of veneer string. The string was then heated and formed to fit into the already excavated channel made for it, as Latta describes in Ref. [1].

Each piece of veneer string needed for the long side of the table top requires spanning nine cycles of the cosinusoidal function. Each cycle requires a length equaling two half cycles or $2*0.8 = 1.6$ inches for the 0.8 inch approximation to $L(0, 0.75)$. So, the total length needed is approximately $9*1.6 = 14.4$ inches. Of course it's best to use a "too long and trim" policy than a "too short and weep" one.

The use of calculus can be circumvented by simply and carefully laying a piece of string (Def. 1 of *string*) along the dark blue line in the figure and marking the beginning and end points at $x = 0$ and $x = 0.75$. Pulling the string taut and directly measuring L yields another approximation acquired experimentally. An easy alternative is to stick clear tape directly on the undulating edge of the pattern, mark the peak and valley locations on the tape, remove the tape from the pattern, then stretch it out and paste the marked tape on a flat surface; the marks directly indicate the locations where the inlay strip needs to be heated and shaped to the cosinusoidal shape. These approaches might be fun for some, but, to paraphrase Irma Rombauer, I prefer the *Joy of Calculus* to get the approximation I used.

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1. Steve Latta, "Dress up your work with creative stringing," *Fine Woodworking*, pp. 56-61, Nov./Dec. 2010.
2. ABS Engraving, 380 Fee Fee Rd., Maryland Heights, MO 63043
3. http://tutorial.math.lamar.edu/Classes/CalcII/ArcLength.aspx
4. https://www.wolframalpha.com/examples/ArcLength.html