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The Compendium*

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Give me the splendid, silent sun, with all his beams full-dazzling.
~ Walt Whitman

* **Compendium**... "giving the sense and substance of the topic within small compass." In dialing, a compendium is a single instrument incorporating a variety of dial types and ancillary tools.

Sundial Design Considerations

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This article grew out of my effort to understand the diptych transmission-dial designed by Fred Sawyer, which he describes in a *Compendium* note written in 1994 [6]. My approach begins with a way to specify the location on a planar surface where the path of a sun's ray striking a point nodus intersects the surface. This intersection locates the shadow of the nodus projected onto the surface. From this, various sundial designs are investigated, including Fred Sawyer's diptych dial. I especially draw upon the book by J. Meeus [1] in this discussion and the article by Fred Sawyer [6].

Table 1. Nomenclature

δ	declination of the sun	δ ranges between -23.45° and $+23.45^\circ$. Declination values are positive (resp. negative) when the sun is above (resp. below) the equatorial plane, reaching $+23.45^\circ$ at the northern solstice (June 21) and -23.45 at the southern solstice (December 21). The declination is 0° at the spring (March 21) and fall (September 23) equinoxes. The sign convention of J. Meeus [1, Ch. 13] is used here.
ϕ	latitude of dial location	Latitude values are positive (resp. negative) for the northern (resp. southern) hemisphere. For example $\phi = +38.65^\circ$ for St. Louis. This is the sign convention used by J. Meeus [1, Ch. 13].
H	local hour angle of the sun	$H = 0^\circ$ when the sun is in the meridian plane of the place (i.e., the sun is directly south), and H is measured positive going westward from south. For example, $$H = \begin{cases} -15^\circ & \text{at 11 hours local apparent time;} \\ 0^\circ & \text{at 12 hours local apparent noon;} \\ +15^\circ & \text{at 13 hours local apparent time.} \end{cases}$$
$R_x(\theta)$	matrix for rotations around the *x*-axis, right-hand rule sign convention[1]	$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
$R_y(\theta)$	matrix for rotations around the *y*-axis, right-hand rule sign convention	$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
$R_z(\theta)$	matrix for rotations around the *z*-axis, right-hand rule sign convention	$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[1] To rotate a vector counterclockwise around the x-axis in a fixed coordinate system, multiply the vector by $R_x(\theta)$. To rotate the coordinate system around the x-axis while keeping a vector in a fixed position, multiply by $R_x^T(\theta) = R_x(-\theta)$. Do this similarly for rotations about other axes.

There will be three coordinate systems of interest. One is an earth-centered system, called the *equatorial system*; another is a local system at the dial location, called the *horizon system*; and the third is a coordinate system in the surface of the sundial, called the *dial system*. Right-hand rule conventions are followed. The earth is considered to be a sphere of radius r_{earth}. The earth's orbit is taken to be a sun-centered circle of radius $r_{earth2sun}$. The origin of the equatorial system is at the center of the earth with rectangular axes (x_e, y_e, z_e), with z_e aligned with the rotation axis of the earth with its positive direction towards the north pole. The (x_e, y_e)-plane is the equatorial plane, and the (y_e, z_e) -plane is aligned with the local meridian plane at the location of the dial. With respect to this system, the sun's angular position rotates clockwise about the z_e-axis, which is the negative direction for a right-hand coordinate system. A vector $\vec{v}$ from the origin to a point at (x, y, z) in this system is

$$\vec{v} = \vec{v}(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = |\vec{v}| \begin{bmatrix} \sin\theta\cos\omega \\ \sin\theta\sin\omega \\ \cos\theta \end{bmatrix}, \tag{1}$$

where $|\vec{v}|$ is the distance of the point from the earth's center, $\theta = \arccos(z/|\vec{v}|)$, with $0^\circ \le \theta \le 180^\circ$, is the angle between the z_e -axis (polar axis) and the vector to the point, and $\omega = \arctan 2(y/|\vec{v}|, x/|\vec{v}|)$, with $0^\circ \le \omega < 360^\circ$, is the angle between the x_e -axis and the projection of $\vec{v}$ onto the (x_e, y_e) -plane (equatorial plane). The function arctan2(*y*,*x*) is the version of arctan(*y*,*x*) that preserves quadrant identity. The representation of $\vec{v}$ in terms of $(|\vec{v}|, \omega, \theta)$ is the standard polar representation in a rectangular coordinate system following the right-hand convention.

Let $\vec{s}_e(H, \delta)$ be the vector in this equatorial system that extends from the earth's center to the sun's center when the sun's hour angle is H and its declination is δ. In the polar representation, $\theta = 270^\circ - H$ and $\omega = 90^\circ - \delta$. Then, $\vec{s}_e(0,0)$ points towards the sun when it lies in the meridian plane (i.e., at solar noon, $H = 0^\circ$) and the equatorial plane (i.e., on an equinox, $\delta = 0$). Then

$$\vec{s}_e(0,0) = r_{earth2sun} \begin{bmatrix} \sin 270^\circ \cos 90^\circ \\ \sin 270^\circ \sin 90^\circ \\ \cos 270^\circ \end{bmatrix} = r_{earth2sun} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}. \tag{2}$$

At solar noon on days when the sun's declination is not zero, this vector becomes

$$s_e(0,\delta) = R_x(-\delta)\vec{s}_e(0,0) = r_{earth2sun} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\delta) & -\sin(-\delta) \\ 0 & \sin(-\delta) & \cos(-\delta) \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = r_{earth2sun} \begin{bmatrix} 0 \\ -\cos\delta \\ \sin\delta \end{bmatrix}. \tag{3}$$

This is also obtained from (1) using $\theta = 90^\circ - \delta$ and $\omega = 270^\circ$. The vector to the sun, $\vec{s}_e(H, \delta)$, is then

$$\vec{s}_e(H,\delta)=R_z(-H)R_x(-\delta)\vec{s}_e(0,0)=r_{earth2sun}R_z(-H)\begin{bmatrix}0\\-\cos\delta\\\sin\delta\end{bmatrix}=r_{earth2sun}\begin{bmatrix}-\sin H\cos\delta\\-\cos H\cos\delta\\\sin\delta\end{bmatrix}. \tag{4}$$

Consider next, a horizon coordinate-system. The origin of this rectangular system lies in the meridian plane at the location of the sundial; so, the origin is on the surface of the earth at the longitude and latitude of the dial. The (x_h, y_h)-plane of this system is tangent to the earth, called the horizon plane, with the positive x_h -axis towards the east and the y_h-axis towards the north. The z_h-axis is towards the zenith at the location. Changing coordinates from the earth-centered system to the horizon system involves rotating the equatorial plane into the horizon plane and translating the origin of the earth center to the place of the dial. In the equatorial coordinate system, the vector from the earth's center to a location on the surface that lies in the meridian plane at latitude ϕ is

$$r_{earth}\begin{bmatrix}0\\\cos\phi\\\sin\phi\end{bmatrix}, \tag{5}$$

so the vector in the equatorial system from the place to the sun is

$$\vec{s}_e(\phi,H,\delta)=\vec{s}_e(H,\delta)-r_{earth}\begin{bmatrix}0\\\cos\phi\\\sin\phi\end{bmatrix}. \tag{6}$$

Consider this vector fixed in space. The horizon coordinate system is obtained from the equatorial system through a coordinate system rotation clockwise about the x_e-axis through an angle of $\bar{\phi}=90^\circ-\phi$ degrees. Let $\vec{s}_h(\phi,H,\delta)$ be this vector represented now in the horizon coordinate system. It is given by

$$\vec{s}_h(\phi,H,\delta)=R_x(-\bar{\phi})\vec{s}_e(\phi,H,\delta)=R_x(-\bar{\phi})\vec{s}_e(H,\delta)-r_{earth}R_x(-\bar{\phi})\begin{bmatrix}0\\\cos\phi\\\sin\phi\end{bmatrix}. \tag{7}$$

Thus,

$$\vec{s}_h(\phi,H,\delta)=r_{earth2sun}\begin{bmatrix}-\sin H\cos\delta\\-\sin\phi\cos H\cos\delta+\cos\phi\sin\delta\\\cos\phi\cos H\cos\delta+\sin\phi\sin\delta\end{bmatrix}-r_{earth}\begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{8}$$

Let $\vec{u}_h(\phi,t,\delta)$ be the unit vector pointed towards the sun in the horizon system. This is

$$\vec{u}_h(\phi,H,\delta)=\frac{\vec{s}_h(\phi,H,\delta)}{|\vec{s}_h(\phi,H,\delta)|}=\frac{1}{\sqrt{1+a^2}}\begin{bmatrix}-\sin H\cos\delta\\-\sin\phi\cos H\cos\delta+\cos\phi\sin\delta\\\cos\phi\cos H\cos\delta+\sin\phi\sin\delta\end{bmatrix}-\frac{a}{\sqrt{1+a^2}}\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{9}$$

where $a = r_{earth} / r_{earth2sun}$. Since $r_{earth2sun} << r_{earth}$, this unit vector is given to a very good approximation by

$$\vec{u}_h(\phi, H, \delta) \cong \begin{bmatrix} -\sin H \cos\delta \\ -\sin\phi \cos H \cos\delta + \cos\phi \sin\delta \\ \cos\phi \cos H \cos\delta + \sin\phi \sin\delta \end{bmatrix} = \cos\delta \begin{bmatrix} -\sin H \\ -\sin\phi \cos H + \cos\phi \tan\delta \\ \cos\phi \cos H + \sin\phi \tan\delta \end{bmatrix}. \qquad (10)$$

When the z_h-component of $\vec{u}_h(\phi, H, \delta)$ is negative, the sun is below the horizon plane. This occurs for hour angles H corresponding to times from sunset to sunrise. The sun is at or above the horizon plane for hour angles that satisfy

$$\cos H \geq -\tan\phi \tan\delta \ , \qquad (11)$$

with equality at sunrise and sunset.

Let $\vec{v}_h = \begin{bmatrix} x_n & y_n & z_n \end{bmatrix}^T$ be a vector in the horizon system that extends from the origin to a point nodus at (x_n, y_n, z_n), so the nodus lies at a height of z_n above the horizon plane at (x_n, y_n). A vector $\vec{\ell}_h$ to any position along a line between the sun and the point nodus is given parametrically by

$$\vec{\ell}_h(x_n, y_n, z_n; \phi, H, \delta : \lambda) = \vec{v}_h + \lambda \vec{u}_h(\phi, H, \delta). \qquad (12)$$

This defines a ray of light from the sun that strikes the point nodus and continues on to form a shadow of the nodus wherever the line strikes a dial surface. Design of a planar sundial then proceeds by defining a coordinate system in the plane of the dial and identifying the location where $\vec{\ell}_h$ strikes the dial plane for various choices of hour angle, H, and declination, δ.

Example 1: horizontal sundial

Let $\vec{e}_{z_h} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ be a unit vector along the positive z_h-axis of the horizon system. Suppose that H is such that

$$\vec{e}_{z_h} \cdot \vec{u}_h(\phi, H, \delta) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \vec{u}_h(\phi, H, \delta) = \cos\phi \cos H \cos\delta + \sin\phi \sin\delta > 0, \qquad (13)$$

corresponding to times when the sun is above the horizon plane and can illuminate the dial surface; these are times such that $\cos H > -\tan\phi \tan\delta$. Consider a planar sundial with its surface in the horizon plane and with a point nodus at (x_n, y_n, z_n). The horizon coordinate system is then the obvious choice of coordinate system to use for the dial itself. The shadow of the nodus falls at the location in the horizon plane where the parameter λ in (12) is such that $\vec{\ell}_h$ intersect the horizon plane. This is the λ in (12) such that $\vec{e}_{z_h} \cdot \vec{\ell}_h = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \vec{\ell}_h = 0$, yielding

$$\lambda = -\frac{\vec{e}_{z_h} \cdot \vec{v}_h}{\vec{e}_{z_h} \cdot \vec{u}_h(\phi, H, \delta)} = -\frac{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \vec{v}_h}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \vec{u}_h(\phi, H, \delta)} = -\frac{z_n}{\cos\phi \cos H \cos\delta + \sin\phi \sin\delta}, \quad (14)$$

where z_n is the height of the nodus above the horizon plane (dial plane). For hour angles such that $\cos H > -\tan\phi \tan\delta$, the shadow of the point nodus is located in the dial plane at

$$(x, y) = \left(x_n + z_n \frac{\sin H \cos\delta}{\cos\phi \cos H \cos\delta + \sin\phi \sin\delta}, y_n + z_n \frac{\sin\phi \cos H \cos\delta - \cos\phi \sin\delta}{\cos\phi \cos H \cos\delta + \sin\phi \sin\delta} \right). \quad (15)$$

For hour angles such that $\cos H < -\tan\phi \tan\delta$, there is no shadow point because the sun lies below the horizon plane. Minor manipulation yields the alternative expression

$$(x, y) = \left(x_n + z_n \frac{\sin H}{\cos\phi \cos H + \sin\phi \tan\delta}, y_n + z_n \frac{\sin\phi \cos H - \cos\phi \tan\delta}{\cos\phi \cos H + \sin\phi \tan\delta} \right). \quad (16)$$

hour lines for a horizontal dial - For each hour angle, H, at a fixed location (i.e., the latitude ϕ is fixed at the site of the dial), equation (15) yields a locus of points as the sun's declination δ varies. A line for the hour corresponding to H is obtained by connecting these points. Asymptotically as $|\tan\delta|$ becomes large, the shadow point at (x, y) tends lines towards the location on the horizon plane at (x_C, y_C), where

$$(x_C, y_C) = \left(x_n, y_n - z_n \frac{1}{\tan\phi} \right). \quad (17)$$

This asymptotic point (x_C, y_C) where all hour lines converge is the *center of the dial*. Consider a line between the center of the dial $(x_n, y_n - z_n / \tan\phi)$ and a point (x, y) defined by (16). Let θ be the angle between this line and the positive y_h-axis. The tangent of this angle is given by

$$\tan\theta = \frac{x - x_c}{y - y_c} = \frac{x - x_n}{y - y_n + z_n / \tan\phi}. \quad (18)$$

After substitution of x and y from (16) and some manipulation, this reduces to the well known expression

$$\tan\theta = \sin\phi \tan H. \quad (19)$$

Thus, the locus of points (x, y) that define the hour line for H all lie on a straight line as the declination δ varies. Therefore, one way to draw the hour line for a given H is to determine (x, y) from (15) for two values of declination (eg., $\pm 23.45^\circ$ for the southern and northern solstices) and connect the two points with a straight line. The shadow of the nodus point moves within this line segment over the course of a year when the declination of the sun ranges through its range of declinations. Extending the line beyond this segment reaches the center of the dial where all extended hour line segments meet.

declination lines for a horizontal dial - For each declination δ at fixed location, (15) yields a locus of points as the hour angle H varies. A line for the declination corresponding to particular choice of δ is obtained by connecting these points. In this way, declination lines for the solstices, equinoxes, and commemorative dates (eg., a birthdate) may be drawn. In doing this, it is usually assumed that the declination changes slowly over a day, and the declination at noon on a particular date is used to represent that date.

Figure 1 displays the hour lines for a horizontal dial for latitude 38.6° (St. Louis). The point nodus is of unit distance above the dial plane at $(x, y) = (0, 0)$.

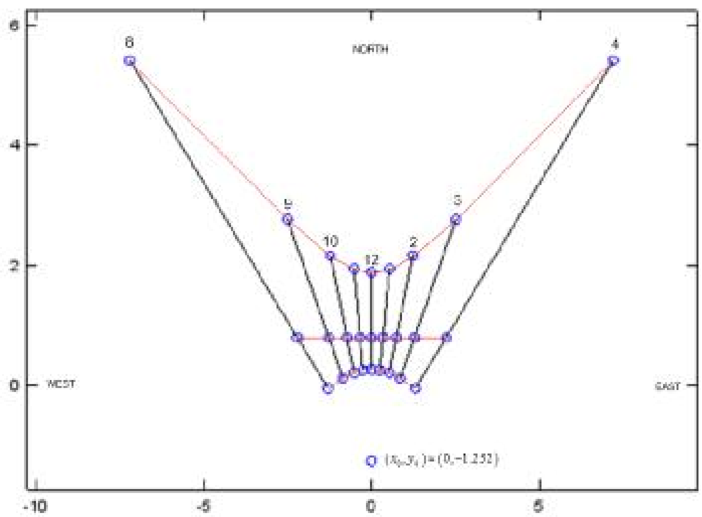

Figure 1. horizontal dial for latitude 38.64° north (St. Louis)

Example 2: general planar sundial

Consider a sundial that lies in a plane, called the *dial plane*. Let $\vec{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$, in horizon coordinates, be a unit vector that is normal to the dial plane. Also, let $\vec{e}_{x_h} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\vec{e}_{y_h} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $\vec{e}_{z_h} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ be unit vectors aligned with the axes of the horizon coordinate system. The orientation of the dial plane can be specified by either the unit vector $\vec{u}$ or, alternatively, by two angles called the *inclination* and *declination* of the plane. The inclination angle, θ_{inc}, with $0 \le \theta_{inc} < 180^\circ$, is the angle between $\vec{u}$ and the unit vector $\vec{e}_{z_h}$ pointing towards the zenith at the location of the dial. This is given by

$$\theta_{inc} = \arccos\left(\frac{\vec{e}_{z_h} \cdot \vec{u}}{\left|\vec{e}_{z_h}\right|\left|\vec{u}\right|}\right) = arc\cos\left(u_z\right) . \tag{20}$$

The declination angle, θ_{dec}, with $-90^{\circ}\left(\text{east of south}\right) \le \theta_{dec} \le 90^{\circ}\left(\text{west of south}\right)$, is the angle between the vector that is the projection of $\vec{u}$ onto the horizon plane and the unit vector, $-\vec{e}_{y_c}$, towards south in the local meridian plane. This is given by

$$\theta_{dec} = \begin{cases} -90^{\circ} - \arctan\left(u_y / u_x\right), & u_x > 0 \\ +90^{\circ} - \arctan\left(u_y / u_x\right), & u_x < 0, u_y \ge 0 \\ +90^{\circ} - \arctan\left(u_y / u_x\right), & u_x < 0, u_y < 0 \\ +180^{\circ}, & u_x = 0, u_y > 0 \\ 0, & u_x = 0, u_y < 0 \\ undefined, & u_x = 0, u_y = 0. \end{cases} . \tag{21}$$

The unit normal to the dial plane can be expressed in terms of the inclination and declination angles as

$$\vec{u} = \begin{bmatrix} -\sin\theta_{inc}\sin\theta_{dec} \\ -\sin\theta_{inc}\cos\theta_{dec} \\ \cos\theta_{inc} \end{bmatrix} . \tag{22}$$

Examples of these angles are given in Table 2 .

Table 2. dial plane specification

dial plane direction	unit normal $\vec{u}$	inclination angle θ_{inc} (degrees)	declination angle θ_{dec} (degrees)
horizontal	$\left[0 \quad 0 \quad 1\right]^T$	0	undefined
vertical facing South	$\left[0 \quad -1 \quad 0\right]^T$	90	0
vertical facing East	$\left[1 \quad 0 \quad 0\right]^T$	90	-90
vertical facing West	$\left[-1 \quad 0 \quad 0\right]^T$	90	90
vertical facing southwest at 45°	$\left[1 \quad 1 \quad 0\right]^T \frac{1}{\sqrt{2}}$	90	45

Consider a right-hand coordinate system in a plane that is coincident with the surface of the planar dial. The axes of this coordinate system are obtained by rotation of the axes of the horizon system, keeping the normal vector $\vec{u}$ of (22) fixed in space. The normal vector $\vec{u}$ defines the *z*-axis o the dial coordinate system. The (x,z)-plane of the dial system is the meridian plane at the dial location, and the (x,y)-plane is the dial plane. The vector $\vec{u}_h(\phi,H,\delta)$ of (10) is expressed in dial coordinates by applying rotations according to

$$\vec{u}_d(\phi,H,\delta,\theta_{dec},\theta_{inc}) = R_x(-\theta_{inc})R_z(\theta_{dec})\vec{u}_h(\phi,H,\delta). \tag{23}$$

Using (10) and

$$R_x(-\theta_{inc})R_z(\theta_{dec}) = \begin{bmatrix} \cos\theta_{dec} & -\sin\theta_{dec} & 0 \\ \cos\theta_{inc}\sin\theta_{dec} & \cos\theta_{inc}\cos\theta_{dec} & \sin\theta_{inc} \\ -\sin\theta_{inc}\sin\theta_{dec} & -\sin\theta_{inc}\cos\theta_{dec} & \cos\theta_{inc} \end{bmatrix} \tag{24}$$

in (23) yields

$$\vec{u}_d(\phi,H,\delta,\theta_{dec},\theta_{inc}) = \cos\delta \begin{bmatrix} \{-\cos\theta_{dec}\sin H + \sin\theta_{dec}\sin\phi\cos H - \sin\theta_{dec}\cos\phi\tan\delta\} \\ \\ \{-\cos\theta_{inc}\sin\theta_{dec}\sin H - \cos\theta_{inc}\cos\theta s_{dec}\sin\phi\cos H \\ \quad +\cos_{inc}\cos\theta_{dec}\cos\phi\tan\delta + \sin\theta_{inc}\cos\phi\cos H \\ \qquad +\sin\theta_{inc}\sin\phi\tan\delta\} \\ \\ \{\sin\theta_{inc}\sin\theta_{dec}\sin H + \sin\theta_{inc}\cos\theta_{dec}\sin\phi\cos H \\ \quad -\sin\theta_{inc}\cos\theta_{dec}\cos\phi\tan\delta + \cos\theta_{inc}\cos\phi\cos H \\ \qquad +\cos\theta_{inc}\sin\phi\tan\delta\} \end{bmatrix} \tag{25}$$

Note that the sun illuminates the dial plane for hour angles, H , such that the sun is above the horizon plane ($\cos H > -\tan\phi\tan\delta$) and on the face side of the dial $\vec{e}_{z_d}\cdot\vec{u}_d(\phi,H,\delta,\theta_{dec},\theta_{inc}) > 0$. Let $\vec{v}_d = [x_n \quad y_n \quad z_n]^T$ denote the vector in dial coordinates that extends from the origin to a point nodus located at height z_n above the dial plane at location (x_n,y_n). All points along a ray of sunlight that strikes the point nodus are on a line given parametrically by

$$\vec{\ell}_d(x_n,y_n,z_n;\phi,H,\delta,\theta_{dec},\theta_{inc}:\lambda) = \vec{v}_d + \lambda\vec{u}_d(\phi,H,\delta,\theta_{dec},\theta_{inc}). \tag{26}$$

The shadow of the nodus falls at the location in the dial plane where the parameter λ in (26) is such that $\vec{\ell}_d$ intersect the dial plane. This is the λ such that $\vec{e}_{z_d}\cdot\vec{\ell}_d = [0 \quad 0 \quad 1]\vec{\ell}_d = 0$, yielding

$$\lambda = -\frac{\vec{e}_{z_d} \cdot \vec{v}_d}{\vec{e}_{z_d} \cdot \vec{u}_d\left(\phi, H, \delta, \theta_{dec}, \theta_{inc}\right)} = -\frac{\begin{bmatrix}0 & 0 & 1\end{bmatrix}\vec{v}_d}{\begin{bmatrix}0 & 0 & 1\end{bmatrix}\vec{u}_d\left(\phi, H, \delta, \theta_{dec}, \theta_{inc}\right)} = -\frac{z_n}{D\cos\delta}, \tag{27}$$

where the denominator, D, is given by

$$\begin{aligned} D = \{&\sin\theta_{inc}\sin\theta_{dec}\sin H + \sin\theta_{inc}\cos\theta_{dec}\sin\phi\cos H \\ &-\sin\theta_{inc}\cos\theta_{dec}\cos\phi\tan\delta + \cos\theta_{inc}\cos\phi\cos H \\ &+\cos\theta_{inc}\sin\phi\tan\delta\}. \end{aligned} \tag{28}$$

The shadow of the point nodus is located at (x, y) in the dial plane. These coordinates are determined from (25) and (26) with λ equal to its value in (27), yielding

$$x = x_n + \frac{z_n}{D}\left\{\cos\theta_{dec}\sin H - \sin\theta_{dec}\sin\phi\cos H + \sin\theta_{dec}\cos\phi\tan\delta\right\} \tag{29}$$

and

$$\begin{aligned} y = y_n + \frac{z_n}{D}\{&\cos\theta_{inc}\sin\theta_{dec}\sin H + \cos\theta_{inc}\cos\theta_{dec}\sin\phi\cos H \\ &-\cos\theta_{inc}\cos\theta_{dec}\cos\phi\tan\delta - \sin\theta_{inc}\cos\phi\cos H \\ &-\sin\theta_{inc}\sin\phi\tan\delta\}. \end{aligned} \tag{30}$$

These coordinates locate the shadow of the point nodus provided the sun shines on the dial plane, which occurs for hour angles, H, that satisfy the two conditions (11) and $D > 0$, where D is given in (28).

hour lines for a planar dial that inclines and declines - For each hour angle, H, at a fixed location and dial orientation (i.e., the latitude, ϕ, dial inclination, θ_{inc}, and dial declination, θ_{dec} are fixed), equations (29) and (30) yield a locus of points as the sun's declination δ varies. A line for the hour corresponding to H is obtained by connecting these points. Asymptotically as $|\tan\delta|$ becomes large, the shadow point at (x, y) tends lines towards the location on the dial plane at (x_C, y_C), where

$$x_c = x_n + z_n \frac{\sin\theta_{dec}\cos\phi}{\cos\theta_{inc}\sin\phi - \sin\theta_{inc}\cos\theta_{dec}\cos\phi} \tag{31}$$

and

$$y_c = y_n - z_n \frac{\sin\theta_{inc}\sin\phi + \cos\theta_{inc}\cos\theta_{dec}\cos\phi}{\cos\theta_{inc}\sin\phi - \sin\theta_{inc}\cos\theta_{dec}\cos\phi} \tag{32}$$

This asymptotic point (x_C, y_C) where all hour lines converge is the *center of the dial*. Consider a line between the center of the dial and a point (x, y) defined by (29) and (30). Let θ be the angle between this line and the positive y_d -axis. The tangent of this angle is given by

$$\tan\theta = \frac{x - x_c}{y - y_c} \tag{33}$$

Substituting (29), (30), (31), and (32) into this expression and simplifying yields

$$\tan\theta = \frac{(\cos\theta_{inc}\cos\theta_{dec}\sin\phi - \sin\theta_{inc}\cos\phi)\tan H - \cos\theta_{inc}\sin\theta_{dec}}{\cos\theta_{dec} + \sin\theta_{dec}\sin\phi\tan H} \quad . \tag{34}$$

Thus, the locus of points (x, y) that define the hour line for H all lie on this straight line as the declination δ varies. Therefore, as with a horizontal dial, one way to draw the hour line for a given H is to determine (x, y) from (29) and (30) for two values of declination (eg., $\pm 23.45^\circ$ for the southern and northern solstices) and connect the two points with a straight line. The shadow of the nodus point moves within this line segment over the course of a year when the declination of the sun ranges through its range of declinations. Extending the line beyond this segment reaches the center of the dial where all extended hour line segments meet. A second way to draw the hour line for a given H is to locate the center of the dial from (31) and (32); the hour line then extends from the dial center at an angle θ , given by (34), to the y_d -axis.

declination lines for a planar dial that inclines and declines - For each declination δ at fixed location, (29) and (30) yield a locus of points as the hour angle H varies. A line for the declination corresponding to a particular choice of δ is obtained by connecting these points. In this way, declination lines for the solstices, equinoxes, and commemorative dates (eg., a birthdate) may be drawn. In doing this, it is usually assumed that the declination changes slowly over a day, and the declination at noon on a particular date is used to represent that date.

Example 3: vertical sundial

The inclination angle, θ_{inc} , for a vertical dial is 90° . The declination angle, θ_{dec} , is determined by the direction the surface of the dial faces, measured from the meridian plane at the dial location, with $\theta_{dec} = -90^\circ$ (equivalently, $+270^\circ$) for an east facing dial, $\theta_{dec} = 0^\circ$ for a south facing dial, and $\theta_{dec} = +90^\circ$ for a west facing dial. From (29) and (30), with $\theta_{inc} = 90^\circ$, the coordinates of the shadow of a point nodus at (x_n, y_n, z_n) are

$$x = x_n + z_n \frac{\cos\theta_{dec}\sin H - \sin\theta_{dec}\sin\phi\cos H + \sin\theta_{dec}\cos\phi\tan\delta}{\sin\theta_{dec}\sin H + \cos\theta_{dec}\sin\phi\cos H - \cos\theta_{dec}\cos\phi\tan\delta} \tag{35}$$

and

$$y = y_n - z_n \frac{\cos\phi \cos H + \sin\phi \tan\delta}{\sin\theta_{dec} \sin H + \cos\theta_{dec} \sin\phi \cos H - \cos\theta_{dec} \cos\phi \tan\delta} \quad . \tag{36}$$

The coordinates (35) and (36) are valid for hour angles that satisfy the condition $\cos H > -\tan\phi \tan\delta$ from (11); these hour angles also satisfy $D > 0$ when $\theta_{inc} = 90^\circ$ for a vertical dial. The location of the center of the dial, (x_c, y_c), is

$$(x_c, y_c) = \left(x_n - z_n \tan\theta_{dec}, y_n + z_n \frac{\tan\phi}{\cos\theta_{dec}} \right) . \tag{37}$$

From (34) the tangent of the angle, θ, that an hour line makes with the y_d-axis is given by

$$\tan\theta = -\frac{\cos\phi \tan H}{\cos\theta_{dec} + \sin\theta_{dec} \sin\phi \tan H} \quad . \tag{38}$$

Figure 2 displays a south facing vertical dial for St. Louis. The point nodus is of unit distance above the dial plane at $(x, y) = (0, 0)$.

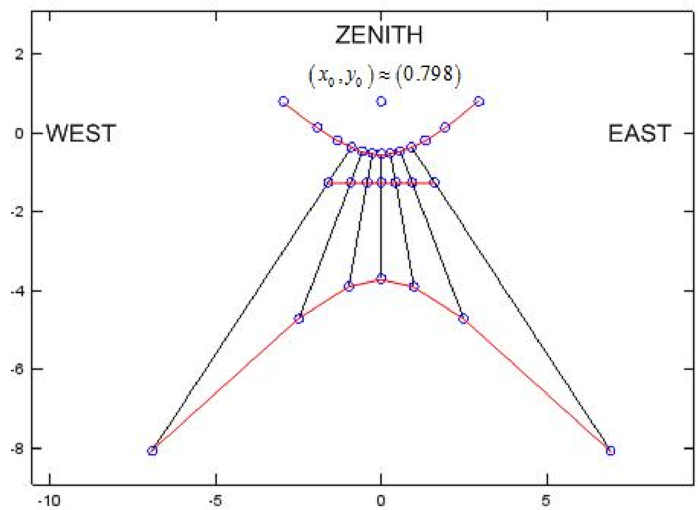

Figure 2. South facing vertical dial for latitude 38.64° north (St. Louis)

Example 4: transmission dial

Consider next the transmission dials described first by Thibaud Taudin-Chabot [5] and Fred Sawyer [6]. These are vertical or inclining-declining dials having hour and declination lines that are drawn on a clear planar surface such that shadows of the lines are produced by transmission of sunlight through the clear surface onto a horizontal surface where times are read. As the sun progresses through hour angles over the course of a day, the hour lines on the dial plane move as shadows across the horizontal surface on which they are projected. Time is indicated by hour lines as they progressively intersect a fixed reference point on the horizontal surface. Because the projection of the dial is being read instead of the dial itself, the dial must be drawn on the dial plane inverted from its orientation when read by looking directly at the dial plate. This amounts to a rotation of the dial coordinate system by 180° degrees around the z_d -axis. Setting $x_n = y_n = 0$ in (35) and (36), the coordinates of points along hour and declination lines become

$$x = -z_n \frac{\cos\theta_{dec} \sin H - \sin\theta_{dec} \sin\phi \cos H + \sin\theta_{dec} \cos\phi \tan\delta}{\sin\theta_{dec} \sin H + \cos\theta_{dec} \sin\phi \cos H - \cos\theta_{dec} \cos\phi \tan\delta} \tag{39}$$

and

$$y = +z_n \frac{\cos\phi \cos H + \sin\phi \tan\delta}{\sin\theta_{dec} \sin H + \cos\theta_{dec} \sin\phi \cos H - \cos\theta_{dec} \cos\phi \tan\delta} \tag{40}$$

The reference point on the horizontal surface now replaces the point nodus, so it is located at $\vec{v}_n = \begin{bmatrix} 0 & 0 & -z_n \end{bmatrix}^T$. The projection dial of Thibaud Taudin-Chabot [5] results by selecting $\theta_{dec} = 0^\circ$ corresponding to a south facing vertical dial. Then

$$\left(x, y\right) = \left(-z_n \frac{\sin H}{\sin\phi \cos H - \cos\phi \tan\delta}, y = +z_n \frac{\cos H + \tan\phi \tan\delta}{\tan\phi \cos H - \tan\delta} \right) \tag{41}$$

The dial of Fred Sawyer [6] is a somewhat more complicated. For it, we must consider two vertical dials, one drawn for declination $\theta_{dec} = 45^\circ$ and the other for $\theta_{dec} = -45^\circ$. These dials are truncated along their meridian lines, $H = 0^\circ$, and then the halves are joined at a 90° angle along the meridian lines. The coordinates of shadow points for the dial at declination angle $\theta_{dec} = +45^\circ$ are

$$\left(x_{+45^\circ}, y_{+45^\circ}\right) = \left(-z_n \frac{\sin H - \sin\phi \cos H + \cos\phi \tan\delta}{\sin H + \sin\phi \cos H - \cos\phi \tan\delta}, \sqrt{2} z_n \frac{\cos\phi \cos H + \sin\phi \tan\delta}{\sin H + \sin\phi \cos H - \cos\phi \tan\delta} \right), \tag{42}$$

and the center of the dial is at $\left(x_c, y_c\right)_{+45^\circ} = \left(-1, \sqrt{2} \tan\phi\right)$. And the coordinates for the dial at declination angle $\theta_{dec} = -45^\circ$ are

$$\left(x_{-45^\circ}, y_{-45^\circ}\right) = \left(-z_n \frac{\sin H + \sin\phi \cos H - \cos\phi \tan\delta}{-\sin H + \sin\phi \cos H - \cos\phi \tan\delta}, \sqrt{2} z_n \frac{\cos\phi \cos H + \sin\phi \tan\delta}{-\sin H + \sin\phi \cos H - \cos\phi \tan\delta} \right), \tag{43}$$

with the center of the dial at $\left(x_c, y_c\right)_{+45^\circ} = \left(+1, \sqrt{2} \tan\phi\right)$. Fred Sawyer's design is, to quote from [6], *Begin by plotting points (x,y) for the afternoon hours on the summer solstice, when the solar declination is* 23.45° *. Similarly, now plot points for the afternoon hours on the winter solstice, when the solar declination is* -23.45°*. Connect the two points for the same hours with a straight line, limiting the line where necessary so it does not extend below the x-axis.*

Now draw lines to connect the successive points for the summer solstice; and continue with a similar set of lines for the winter solstice points.

Mirror the pattern around the noon line so that a similar panel is drawn for the morning hours. The resulting pattern must be folded along the noon line to form two panels which stand at a 90° *angle to each other. This diptych is then placed standing in the sun with the meridian (true north-south line) running down the middle of the angle, bisecting it.*

As a final touch, a single point on the meridian line and between the two panels must be highlighted [as the point where time is read].

This design was followed using (42) and (43) implemented in Octave for the latitude, 38.64° north, of St. Louis MO and a point nodus 1 unit above the dial plane. The resulting design is displayed in Figure 3.

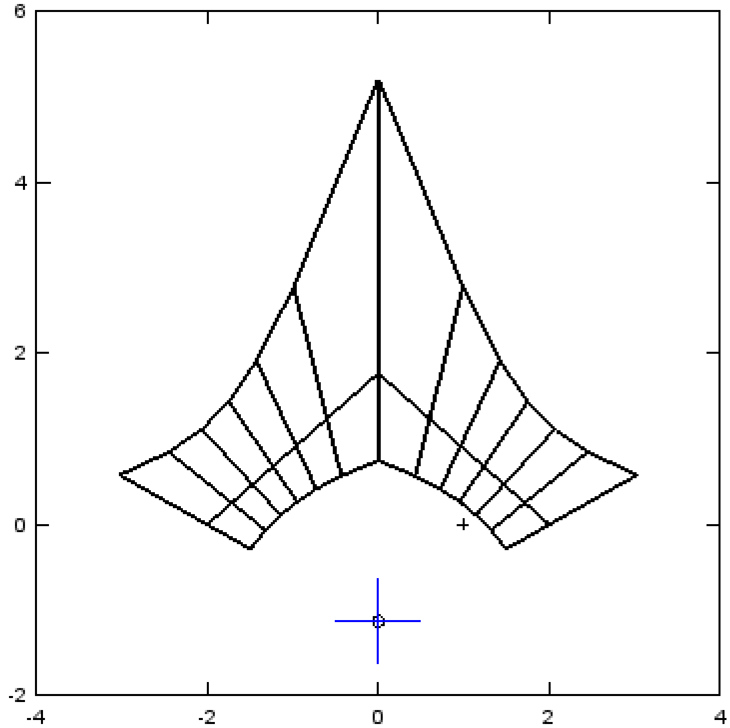

Figure 3. Fred Sawyer's diptych dial design for latitude 38.64 degrees north

The Octave procedure used to produce this drawing is listed in the Appendix. It was called as PlanarDial(38.6443,90,-45,-1,0,1,-1).

The sundial I made that is based on this design is shown in Figure 4. The two glass panels were made by

Figure 4. Projection Dial

Preston Art Glass, located in St. Louis. The hour, solstice, and equinox lines were made using a silk-screen method, and the colored regions were made by painting Reusche & Co.[9] transparent stains (amber and blue) into the regions and then kiln firing the panels at about 1080`F. The wooden base is hard maple, and the legs are made of Australian Figured Red Gum shaped to reflect the St. Louis Arch. A picture of the dial taken at about 3:00PM CST on Dec. 30, 2014 is in Figure 5.

Figure 5. Dial at 3:00PM CST on Dec. 30, 2014

References

1. J. Meeus, *Astronomical Algorithms*, Willmann-Bell Inc., 1991.
2. Harold E. Brandmaier, "Sundial Design Using Matrices," *NASS Compendium*, Vol. 5, No. 3, pp. 12-16, September 1998.
3. Harold E. Brandmaier, "Equatorial Projection Sundial Projection Vectors," *NASS Compendium*, Vol. 12, No. 1, pp. 16-23, March 2005.
4. Herb Ramp, "Designing Declining/Inclining Sundials Using Vectors," *NASS Compendium*, Vol. 13, No. 3, pp. 24-28, September 2006.
5. Thibaud Taudin-Chabot, "A Transparent Window Dial," *NASS Compendium*, Vol. 1, No. 1, p. 6, August 1994.
6. Fred Sawyer, "A stained Glass Diptych Pattern," *NASS Compendium*, Vol. 1, No. 1, pp. 6-8, August 1994.
7. Ortwin Feustel, "The Culmination Sundial," *NASS Compendium*, Vol. 19, No. 4, pp. 26-36, December 2012.
8. GNU Octave programming language http://www.gnu.org/software/octave/.
9. Reusche & Co. http://www.reuscheco.com/.

Appendix. Octave procedure for inclining and declining dial planes

GNU Octave [8] is a high level programming language that is especially suited to numerical calculations involving vectors and matrices. The following Octave procedure implements (28)-(34) and produces preliminary drawings for dials on planar, declining/inclining surfaces.

```
function [DialParameters,denominators,xcoordinates,ycoordinates,xc,yc]= ...
                                PlanarDial(latDIAL,incDIAL,decDIAL,xn,yn,zn,sw)
%   PlanarDial returns the x & y coordinates of hour lines for
%%      three declinations (-23.45 0 23.45) degrees.  For each
%%     declination, the hour line coordinates are given for
%%     hour line angles (-90 -75 -60 -45 -30 -150 15 30 45 60 75 90) degrees.
%% Denominator values are returned for each declination for each hour angle.
%       Coordinates should be ignored for negative denomintor values as they are not valid.
% Inputs:
%%   latDIAL = latitude of the dial location in degrees (+ north, - south)
%%   decDIAL (declination) & incDIAL (inclination) of dial plane in degrees
%%   (xn,yn,zn) are the coordinates of a point nodus distance zn above the dial plane
%%       at (xn,yn) in dial plane coordinates.
%%   sw(itch) = -1 for transmission dial, +1 otherwise
%
% define parameters
d2r = pi/180;                    %factor to convert degrees to radians
lat = latDIAL*d2r;               %latitude of location, radians
decDIAL = decDIAL*d2r;
incDIAL = incDIAL*d2r;
decSUN = [-23.45 0 23.45]*d2r;      %declination of sun at solstices and equinoxes
h = [-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6]*15*d2r;        %hour angles
%
% begin calculations
%% determine dial center coordinates
DenomCenter = cos(incDIAL)*sin(lat) - sin(incDIAL)*cos(decDIAL)*cos(lat);
xcenter = xn + zn*(sin(decDIAL)*cos(lat))/DenomCenter; xcenter = xcenter*sw;
ycenter = yn - zn*(sin(incDIAL)*sin(lat) + cos(incDIAL)*cos(decDIAL)*cos(lat))/DenomCenter;
  ycenter = ycenter*sw;
%
% determine denominator for each sun declination
Denom1 =  sin(incDIAL)*sin(decDIAL)*sin(h) + sin(incDIAL)*cos(decDIAL)*sin(lat)*cos(h) ...
        - sin(incDIAL)*cos(decDIAL)*cos(lat)*tan(decSUN(1)) + cos(incDIAL)*cos(lat)*cos(h) ...
                                + cos(incDIAL)*sin(lat)*tan(decSUN(1));
%
Denom2 =  sin(incDIAL)*sin(decDIAL)*sin(h) + sin(incDIAL)*cos(decDIAL)*sin(lat)*cos(h) ...
        - sin(incDIAL)*cos(decDIAL)*cos(lat)*tan(decSUN(2)) + cos(incDIAL)*cos(lat)*cos(h) ...
                                + cos(incDIAL)*sin(lat)*tan(decSUN(2));
%
Denom3 =  sin(incDIAL)*sin(decDIAL)*sin(h) + sin(incDIAL)*cos(decDIAL)*sin(lat)*cos(h) ...
        - sin(incDIAL)*cos(decDIAL)*cos(lat)*tan(decSUN(3)) + cos(incDIAL)*cos(lat)*cos(h) ...
```

```
                                + cos(incDIAL)*sin(lat)*tan(decSUN(3));
%
% determine x coordinates for each sun declination
x1 = xn + zn*(cos(decDIAL)*sin(h) - sin(decDIAL)*sin(lat)*cos(h) ...
                                + sin(decDIAL)*cos(lat)*tan(decSUN(1)))./Denom1;
x1 = x1*sw;
%
x2 = xn + zn*(cos(decDIAL)*sin(h) - sin(decDIAL)*sin(lat)*cos(h) ...
                                + sin(decDIAL)*cos(lat)*tan(decSUN(2)))./Denom2;
x2 = x2*sw;
%
x3 = xn + zn*(cos(decDIAL)*sin(h) - sin(decDIAL)*sin(lat)*cos(h) ...
                                + sin(decDIAL)*cos(lat)*tan(decSUN(3)))./Denom3;
x3 = x3*sw;
%
% determine y coordinates for each sun declination
y1 = yn + zn*(cos(incDIAL)*sin(decDIAL)*sin(h) + cos(incDIAL)*cos(decDIAL)*sin(lat)*cos(h) ...
                                        - cos(incDIAL)*cos(decDIAL)*cos(lat)*tan(decSUN(1)) ...
                                        - sin(incDIAL)*cos(lat)*cos(h) -
sin(incDIAL)*sin(lat)*tan(decSUN(1)))./Denom1;
y1 = y1*sw;
%
y2 = yn + zn*(cos(incDIAL)*sin(decDIAL)*sin(h) + cos(incDIAL)*cos(decDIAL)*sin(lat)*cos(h) ...
                                        - cos(incDIAL)*cos(decDIAL)*cos(lat)*tan(decSUN(2)) ...
                                        - sin(incDIAL)*cos(lat)*cos(h) -
sin(incDIAL)*sin(lat)*tan(decSUN(2)))./Denom2;
y2 = y2*sw;
%
y3 = yn + zn*(cos(incDIAL)*sin(decDIAL)*sin(h) + cos(incDIAL)*cos(decDIAL)*sin(lat)*cos(h) ...
                                        - cos(incDIAL)*cos(decDIAL)*cos(lat)*tan(decSUN(3)) ...
                                        - sin(incDIAL)*cos(lat)*cos(h) -
sin(incDIAL)*sin(lat)*tan(decSUN(3)))./Denom3;
y3 = y3*sw;
%
% outputs
DialParameters = [incDIAL/d2r decDIAL/d2r xn yn zn sw]
  disp ("dial input parameters [incDIAL decDIAL xn yn zn switch]"), disp (DialParameters), disp(" ")
center = [xcenter ycenter];
  disp ("center of dial [xcenter ycenter] ="), disp(center), disp(" ")
denominators = [Denom1; Denom2; Denom3]
xcoordinates = [x1; x2; x3]
ycoordinates = [y1; y2; y3;]
HourAngles = atand((x1 - xcenter)./(y1 - ycenter))
HourAngles = atand((x2 - xcenter)./(y2 - ycenter))
HourAngles = atand((x3 - xcenter)./(y3 - ycenter))
%
%draw dial
plot(xn*sw,yn*sw,"+k"); hold on; plot(xcenter,ycenter,'ok'); axis equal
```

```
nmin = 7-6; nmax = 7+0;
for n = nmin:nmax
  hold on; plot([x1(n); x3(n)],[y1(n); y3(n)],"k",'linewidth',2)
  hold on; plot(2*xcenter-[x1(n); x3(n)],[y1(n); y3(n)],"k",'linewidth',2)
  endfor
for n = nmin:nmax-1
  hold on; plot([x1(n); x1(n+1)],[y1(n); y1(n+1)],"k",'linewidth',2)
  hold on; plot(2*xcenter-[x1(n); x1(n+1)],[y1(n); y1(n+1)],"k",'linewidth',2)
  hold on; plot([x2(n); x2(n+1)],[y2(n); y2(n+1)],"k",'linewidth',2)
  hold on; plot(2*xcenter-[x2(n); x2(n+1)],[y2(n); y2(n+1)],"k",'linewidth',2)
  hold on; plot([x3(n); x3(n+1)],[y3(n); y3(n+1)],"k",'linewidth',2)
  hold on; plot(2*xcenter-[x3(n); x3(n+1)],[y3(n); y3(n+1)],"k",'linewidth',2)
endfor
% add nodus height (zn) scale line at center for reference and scale checking
hold on; plot([xcenter-zn/2 xcenter+zn/2],[ycenter ycenter])
hold on; plot([xcenter xcenter],[ycenter-zn/2 ycenter+zn/2])
%
% save data to a file
%% set precision of saved data
  center = round(center.*1000)./1000;
  denominators = round(denominators.*1000)./1000;
  xcoordinates = round(xcoordinates.*1000)./1000;
  ycoordinates = round(ycoordinates.*1000)./1000;
  HourAngles = round(HourAngles.*100)./100;
%now save the data
save DialData center denominators xcoordinates ycoordinates HourAngles
%
%save plot of dial
print("MyDial.pdf")
print("MyDial2.dxf")
endfunction
```