The Compendium*

Honor the beauty and wisdom of time.

~ *Thomas J.S. Mikelson*

* Compendium... *"giving the sense and substance of the topic within small compass." In dialing, a compendium is a single instrument incorporating a variety of dial types and ancillary tools.*

Polyhedral Sundials
Donald Snyder (St. Louis Mo)

Sundial designers have been placing their dials on the facets of polyhedra for a long time. Several very colorful ones are displayed in the book by Lennox-Boyd [13] that date from the 16th [13, pp. 70-71, 77] and the 19th [13, p. 118-119] centuries. Pictures of many more can be found with a Google search on "polyhedral sundial." Three examples are displayed in Figure 1.
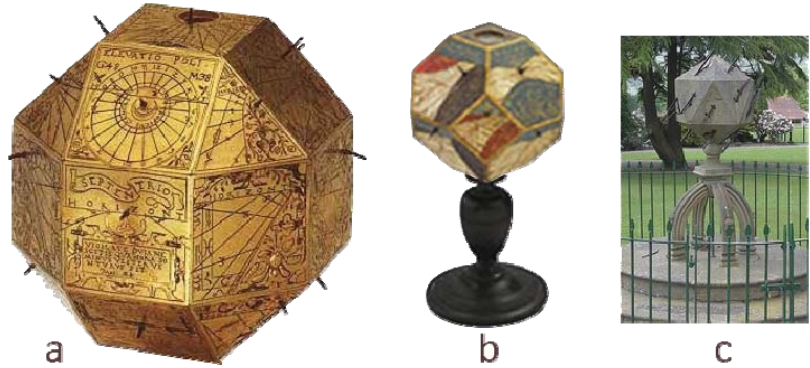

Figure 1. Polyhedral sundials: (a) 16th century, attributed to Hans Roch; (b) 16th century, attributed to Stefano Buonsignori; (c) 19th century, in Marsden Park, Lancashire,UK[1]

A polyhedron is a three dimensional object comprised of multiple, flat facets that are connected to one another along their edges. There are many types of polyhedra. We will be concerned with *convex* polyhedra, but sundials on the surface of nonconvex polyhedra are also possible. A convex polyhedron is one in which a line connecting any two points on the surface either lies on the surface or passes entirely inside the polyhedron. A *regular* polyhedron is one in which all facets are identical regular polygons and each vertex has the same configuration of facets as the others. It has been known since the days of Plato that there are only five regular polyhedra. These are the *Platonic solids* diagrammed in Figure 2.
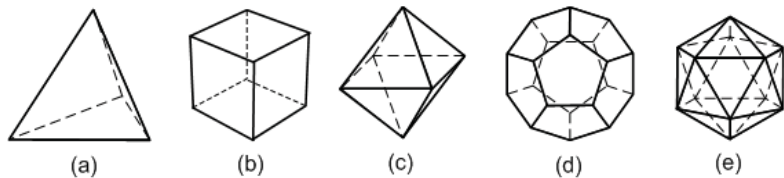

Figure 2. The five Platonic polyhedra: (a) tetrahedron, (b) cube, (c) octahedron, (d)dodecahedron, (e) icosahedron

The *Archimedean solids* are also convex polyhedra. They are not regular; while the facets are regular polygons, the facets in any one of them are not all the same. There are thirteen Archimedean solids, shown in Figure 3.
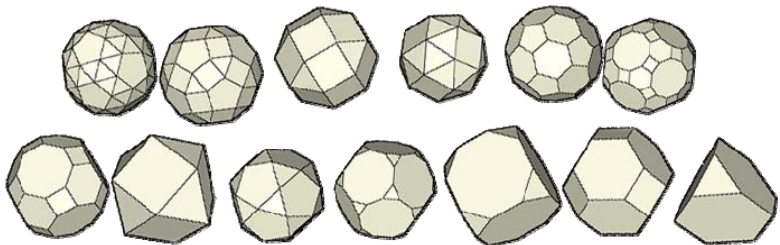
Polyhedral dials (a) and (b) of Figure 1 are the Archimedean types *rhombicuboctahedron* and *truncated octahedron*, respectively, and (c) is the Platonic type *icosahedron*. Polyhedral shapes are seen not only in sundial constructions but

Figure 3. The thirteen Archimedean solids

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[1] For (c): original image http://www.flickr.com/photos/rossendalewadey/5736931333 copyright R. Wade. This work licensed under the Creative Commons Attribution-Share Alike 2.0 Generic. To view a copy of this license, visit http://creativecommons.org/licenses/by-sa/2.0/.

in various branches of science as well. For example, the truncated octahedron occurs in the study of energy bands of solid-state materials[2], and the icosahedron of dial (c) is seen in bactriophage lambda (a virus that infects bacteria)[3]. There are many more polyhedra besides the Platonic and Archimedean ones (Cromwell [1] has examples), and some have been used for sundials.

Three spheres having a common center are associated with each of the Platonic and Archimedean solids. These are: a *circumscribed sphere*, which surrounds the solid, touching all vertices; a *midsphere*, which is tangent to all edges; and, an *inscribed sphere*, or *insphere*, that is tangent to all facets.

The design of a polyhedral sundial proceeds in several steps:

a. choose the polyhedron to be used;
b. choose the orientation of the polyhedron;
c. determine the inclination and declination of the facets that are to support sundials;
d. select the type(s) of sundials to be used;
e. design the sundials;
f. complete the construction.

There are many options in choosing the orientation in Step b. A natural choice is to select one facet to rest on a horizontal surface and then rotate the polyhedron on that 'bottom' facet until a plane of reflection symmetry for the polyhedron lies in the local meridian at the location where the sundial is to be placed. This is not a necessary choice but is the one to be explored. A way to determine the inclination and declination of a facet in Step c is first to determine the unit normal vector to the plane containing the facet and from that the inclination and declination. Various types of sundials can be considered for Step d, including polar gnomon, pole gnomon, bifilar, and others. Designing the sundials can proceed in many ways, including empirical methods (see Rohr [15, Ch. 3] and Waugh [16, Ch. 12]), graphical methods (see Mayall and Mayall [14, pp. 122-125], Rohr [15, Ch. 3]), dialing-scale methods described by Sturmy[4] [11], mathematical methods [see Horst [12]), and computer-implemented methods (see Casalegno [7], Blateyron [8], Sonderegger [9], de Vries [10], and others.[5]) The determination, in Step c, of the inclination and declination of each of the facets to receive a sundial is addressed after first identifying normal vectors for the facets.

Facet Normal Vectors

Shown in Figure 4 is a dodecahedron with its twenty vertices labeled *a*, *b*, …, *t*. This example of a Platonic polyhedron will be used to illustrate a methodology that can applied for other polyhedra. Establish a rectangular coordinate system having its origin at the center of the dodecahedron; this center is the common center of its circumscribed, mid, and inscribed spheres. The orientation of the twelve facets in this coordinate system can be specified by twelve unit vectors, with each vector being oriented so it is perpendicular to the plane containing a facet. These unit vectors point towards the points of tangency of the insphere and facets. The origin $\vec{o} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ of this coordinate system is at the center of the dodecahedron, with the z-axis penetrating the centers of the facets labeled *k-l-m-n-o* and *p-q-r-s-t*; these two facets are parallel to the *x,y*-plane. The *y,z*-plane contains the four vertices *a*, *f*, *n* and *q*. This plane would be the local meridian plane when designing sundials to be placed on the facets of the dodecahedron in this orientation. It is one of the planes about which the dodecahedron exhibits reflection symmetry; another would be the plane containing the vertices *j*, *k*, *e* and *s*. In the coordinate system and orientation of Figure 5, the unit vector in the direction towards and perpendicular to facet *k-l-m-n-o* is simply the unit

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[2] See http://en.wikipedia.org/wiki/Brillouin_zone.
[3] See http://biology-pictures.blogspot.com/2011/11/bacteriophage-t4-image.html for a scanning electron microscope image showing that bactriophage lambda appears remarkably similar to the dial of Figure 1(c) but only about 300 nm in size. Thanks to R. Snyder for noting this.
[4] Reference provided by F. W. Sawyer III.
[5] Additional software procedures for sundial design are listed by the North American Sundial Society on the *Dial Links/Sundial Software* page of their website http://sundials.org/.

vector $\vec{e}_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ along the *z*-axis. If this unit vector is translated from the origin to the point of tangency of the facet *k-l-m-n-o* and the inscribed sphere, it will then point away from the side of the surface to receive a sundial. Obviously, the unit vector towards and perpendicular to the plane *p-q-r-s-t* is $-\vec{e}_z$.
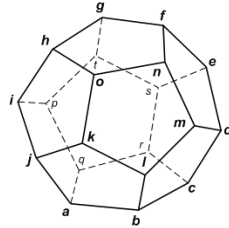

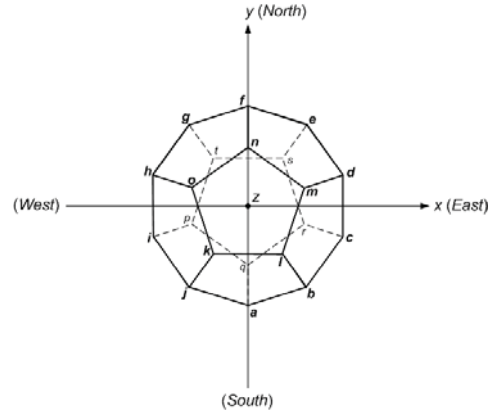
Figure 4. Dodecahedron with vertices labeled

Figure 5. Dodecahedron in coordinate system

The unit vectors towards the other ten facets of the dodecahedron are not aligned with any of the coordinate axes, but they can be identified through rotations of unit vectors along the axes. Consider, for example, the declining facet *a-b-l-k-j*. The unit vector towards this facet and perpendicular to it is obtained by rotating $\vec{e}_y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ in the *y,z*-plane through the angle θ indicated in Figure 6, which shows a portion of the profile of the dodecahedron cut by the *y,z*-plane. For a dodecahedron, the dihedral angle θ_d between facets *k-l-m-n-o* and *a-b-l-k-j* is given by $\theta_d \equiv \arccos\left(-\sqrt{5}/5\right) = \pi - \arctan(2) \approx 116.565°$. The angle $\pi - \theta_d = \arctan(2)$ is the supplement of the dihedral angle, and the angle θ through which $\vec{e}_y$ must be rotated is $\pi/2$ plus this supplement angle, so $\theta = \pi/2 + \arctan(2) \approx 153.435°$. The rotation of $\vec{e}_y$ through the angle θ is counterclockwise around the *x*-axis, so it is positive (i.e., $+\theta$) for the right-hand coordinate system of Figure 5.
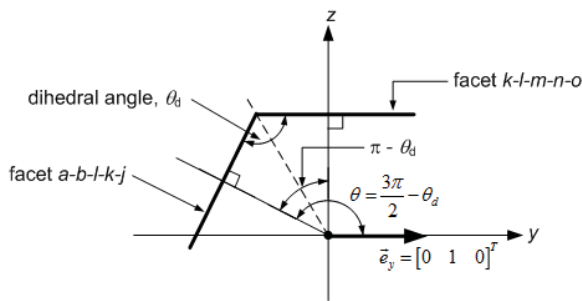

Figure 6. Portion of profile of dodecahedron cut by *x,z*-plane
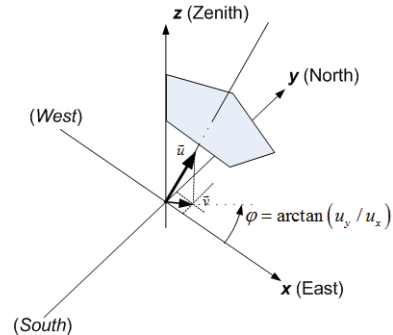
Figure 7. Projection unit normal of planar sundial surface onto the horizon plane

Denote the resulting unit vector by $\vec{u}_{ablkj}$. Then,

$$\vec{u}_{ablkj} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 0 \\ -0.894 \\ 0.447 \end{bmatrix}. \tag{1}$$

The matrix

$$R_x(\varphi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_x & -\sin\varphi_x \\ 0 & \sin\varphi_x & \cos\varphi_x \end{bmatrix}$$

used in this step with $\varphi_x = \theta$ performs the needed rotation about the x-axis. Rotations about the y and z axes are also be useful. These are performed by using the rotation matrices

$$R_y(\varphi_y) = \begin{bmatrix} \cos\varphi_y & 0 & \sin\varphi_y \\ 0 & 1 & 0 \\ -\sin\varphi_y & 0 & \cos\varphi_y \end{bmatrix} \text{ and } R_z(\varphi_z) = \begin{bmatrix} \cos\varphi_z & -\sin\varphi_z & 0 \\ \sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This process is repeated for each facet, so consider next facet b-c-d-m-l. The unit vector for this facet, call it $\vec{u}_{bcdml}$, is obtained from $\vec{u}_{ablkj}$ by a counterclockwise rotation around the z-axis through an angle $+\theta_p = 2\pi/5 = 72°$, the angle subtended at the center of a regular pentagon by any of its edges. Then,

$$\vec{u}_{bcdml} = R_z(\theta_p)\vec{u}_{ablkj} = \begin{bmatrix} -\sin\theta_p\cos\theta \\ \cos\theta_p\cos\theta \\ \sin\theta \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin\theta_p \\ -2\cos\theta_p \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.894\times0.951 \\ -0.894\times0.309 \\ 0.447 \end{bmatrix} = \begin{bmatrix} 0.850 \\ -0.276 \\ 0.447 \end{bmatrix}. \qquad (2)$$

The unit vectors, $\vec{u}_{defnm}$, $\vec{u}_{fghon}$, and $\vec{u}_{hijko}$, towards and perpendicular to facets d-e-f-n-m, f-g-h-o-n, and h-i-j-k-o are obtained from $\vec{u}_{ablkj}$ through rotations about the z-axis by angles $2\theta_p$, $3\theta_p$, and $4\theta_p$, respectively. Then,

$$\vec{u}_{defnm} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin2\theta_p \\ -2\cos2\theta_p \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.526 \\ 0.724 \\ 0.447 \end{bmatrix}, \qquad \vec{u}_{fghon} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin3\theta_p \\ -2\cos3\theta_p \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.526 \\ 0.724 \\ 0.447 \end{bmatrix},$$

and

$$\vec{u}_{hijko} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin4\theta_p \\ -2\cos4\theta_p \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.851 \\ -0.276 \\ 0.447 \end{bmatrix}. \qquad (3)$$

It is evident from Figure 5 that the dodecahedron is a periodic structure with a period of π radians when rotated around the x-axis. As a result, the unit vectors associated with the remaining facets can be obtained by rotating the corresponding unit vectors through an angle of π radians around the x-axis. This rotation is obtained through multiplication of a vector $\vec{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$ by the rotation matrix

$$R_x(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

which results in the vector $\begin{bmatrix} u_x & -u_y & -u_z \end{bmatrix}^T$. Hence, the y and z elements of the vector only need a sign change for this rotation, yielding:

$$\vec{u}_{efgts} = R_x(\pi)\vec{u}_{ablkj} = \begin{bmatrix} 0 \\ -\cos\theta \\ -\sin\theta \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.894 \\ -0.447 \end{bmatrix}, \quad \vec{u}_{ghipt} = R_x(\pi)\vec{u}_{hijko} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin4\theta_p \\ 2\cos4\theta_p \\ -1 \end{bmatrix} \approx \begin{bmatrix} -0.851 \\ 0.276 \\ -0.447 \end{bmatrix},$$

$$\vec{u}_{ijaqp} = R_x(\pi)\vec{u}_{fghon} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin3\theta_p \\ 2\cos3\theta_p \\ -1 \end{bmatrix} \approx \begin{bmatrix} -0.526 \\ -0.724 \\ -0.447 \end{bmatrix}, \quad \vec{u}_{abcrq} = R_x(\pi)\vec{u}_{defnm} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2\sin2\theta_p \\ 2\cos2\theta_p \\ -1 \end{bmatrix} \approx \begin{bmatrix} 0.526 \\ -0.724 \\ -0.447 \end{bmatrix},$$

and $\qquad \vec{u}_{cdesr} = R_x(\pi)\vec{u}_{bcdml} = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 2\sin\theta_p \\ 2\cos\theta_p \\ -1 \end{bmatrix} \approx \begin{bmatrix} 0.850 \\ 0.276 \\ -0.447 \end{bmatrix}.$ $\qquad\qquad$ (4)

Unit vectors for other orientations of the dodecahedron can be obtained in a straightforward way from these ten unit vectors of the particular orientation in Figure 5. This is because other orientations can be specified by rotation angles φ_x, φ_y, and φ_z about the x-axis, y-axis, and z-axis, respectively. If the desired orientation is specified by successive rotations about the axes in this order, then a rotation matrix $R(\varphi_x,\varphi_y,\varphi_z) = R_z(\varphi_z)R_y(\varphi_y)R_x(\varphi_x)$ can be applied, with the unit vector for a facet in the chosen orientation determined from the unit vector $\vec{u}$ for the corresponding facet in the orientation of Figure 5 by the operation $R(\varphi_x,\varphi_y,\varphi_z)\vec{u}$.

Facet Inclination and Declination Angles from Unit Normal Vectors

Shown in Figure 7 is a planar surface on which a sundial may be placed; for illustration, this is facet *d-e-f-n-m* of the dodecahedron in Figure 5. The unit normal vector pointing towards and perpendicular to the surface is $\vec{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$.

<u>Inclination Angle</u> The *inclination* of a planar surface is the angle, denoted by φ_{inc}, between the z-axis (zenith direction at the location) and the vector normal to the surface. Thus,

$$\varphi_{inc} = \arccos\left(\frac{\vec{u}\cdot\vec{e}_z}{|\vec{u}||\vec{e}_z|}\right) = \arccos(u_z). \qquad\qquad (5)$$

<u>Declination Angle</u> The *declination* of a planar surface is an angle denoted by φ_{dec} and defined as follows. First, project the unit normal vector $\vec{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$ onto the x,y-plane (horizon plane). This yields the vector $\vec{v} = \begin{bmatrix} u_x & u_y & 0 \end{bmatrix}^T$ lying in the x,y-plane. Provided $\vec{u}$ is neither perpendicular to the x,y-plane (i.e., not of the form $\begin{bmatrix} 0 & 0 & \pm1 \end{bmatrix}^T = \pm\vec{e}_z$) nor confined to the x,y-plane (i.e., not of the form $\begin{bmatrix} u_x & u_y & 0 \end{bmatrix}^T$ with either $u_x \neq 0$ or $u_y \neq 0$), the vectors $\vec{u}$ and $\vec{v}$ define a vertical plane having a dihedral angle $\varphi = \arctan 2(u_y,u_x)$ with the x,z-plane. This form of arctangent function makes evident the quadrant where the vector $\begin{bmatrix} u_x & u_y \end{bmatrix}^T$ lies. By definition (in radians),

$$\arctan 2(u_y,u_x) = \begin{cases} \arctan(u_y/u_x), & u_x > 0 \\ \pi + \arctan(u_y/u_x), & u_x < 0, u_y \geq 0 \\ -\pi + \arctan(u_y/u_x), & u_x < 0, u_y < 0 \\ +\pi/2, & u_x = 0, u_y > 0 \\ -\pi/2, & u_x = 0, u_y < 0 \\ undefined, & u_x = 0, u_y = 0. \end{cases} \qquad (6)$$

With this definition, the angle $\varphi = \arctan 2(u_y,u_x)$ is measured from the positive x-axis with positive (resp. negative) values corresponding to counterclockwise (resp. clockwise) rotations of $\begin{bmatrix} u_x & u_y \end{bmatrix}^T$ (i.e., right-hand rule) and $-\pi < \varphi \leq \pi$. By traditional definition, the declination angle φ_{dec} of the surface in Figure 7 is the dihedral angle between the y,z-plane (i.e., the local meridian plane) and the plane containing $\vec{u}$ and $\vec{v}$. Further, the declination angle is taken as positive when the normal to the plane

containing $\vec{u}$ and $\vec{v}$ points towards the West of the meridian plane (i.e., $u_x > 0$) and negative when it is towards the East (i.e., $u_x < 0$); this is a left-hand rule. Also, measuring φ_{dec} modulo 2π places the declination angle in the range $-\pi < \varphi_{dec} \le \pi$. Under these conditions,

$$\varphi_{dec} = -\frac{\pi}{2} - \varphi = \begin{cases} -\dfrac{\pi}{2} - \arctan\left(u_y / u_x\right), & u_x > 0 \\[6pt] +\dfrac{\pi}{2} - \arctan\left(u_y / u_x\right), & u_x < 0, u_y \ge 0 \\[6pt] +\dfrac{\pi}{2} - \arctan\left(u_y / u_x\right), & u_x < 0, u_y < 0 \\[6pt] +\pi, & u_x = 0, u_y > 0 \\[3pt] 0, & u_x = 0, u_y < 0 \\[3pt] undefined, & u_x = 0, u_y = 0. \end{cases} \tag{7}$$

Inclination and declination angles are given in Table 1 for some surface orientations.

Table 1. Examples of inclination and declination angles

surface direction	surface normal $\vec{u}$	inclination angle φ_{inc} (degrees)	declination angle φ_{dec} (degrees)
horizontal	$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$	0	undefined
vertical facing South	$\begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T$	90	0
vertical facing East	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$	90	-90
vertical facing West	$\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T$	90	90
inclining-declining as facet *d-e-f-n-m* in Fig. 7	$\begin{bmatrix} 2\sin 144° & -2\cos 144° & 1 \end{bmatrix}^T / \sqrt{5}$	63.435	-144

Example: Inclination and Declination Angles of Facets of a Dodecahedron

We now wish to determine the inclination and declination angles of the facets of the dodecahedron in the orientation of Figure 5. This is the first step in designing sundials on the facets using the sundial-design software, *Sonne*, created by Helmut Sonderegger [9] and available at http://www.helson.at/sun.htm/. The *Sonne* software permits sundials on planar surfaces to be designed. Several parameters need to be specified, including the longitude and latitude of the location where the sundial is to be placed, the orientation of the planar surface in terms of its inclination and declination, and size parameters. The unit vectors and corresponding inclination and declination angles for each of the twelve facets are listed in Table 2.

Table 2. Dodecahedron parameters for orientation in Figure 5.

dodecahedron facet (Figure 5)	unit normal vector $(\theta_p = 72°)$	inclination angle (degrees)	declination angle (degrees)
k-l-m-n-o (top)	$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$	0	Undefined
a-b-l-k-j	$\begin{bmatrix} 0 & -2 & 1 \end{bmatrix}^T / \sqrt{5}$	63.435	0
b-c-d-m-l	$\begin{bmatrix} 2\sin\theta_p & -2\cos\theta_p & 1 \end{bmatrix}^T / \sqrt{5}$	63.435	-72.0
d-e-f-n-m	$\begin{bmatrix} 2\sin 2\theta_p & -2\cos 2\theta_p & 1 \end{bmatrix}^T / \sqrt{5}$	63.435	-144.0

f-g-h-o-n	$\left[\begin{array}{ccc}2\sin3\theta_p & -2\cos3\theta_p & 1\end{array}\right]^T/\sqrt5$	63.435	144.0
h-i-j-k-o	$\left[\begin{array}{ccc}2\sin4\theta_p & -2\cos4\theta_p & 1\end{array}\right]^T/\sqrt5$	63.435	72.0
e-f-g-t-s	$\left[\begin{array}{ccc}0 & 2 & -1\end{array}\right]^T/\sqrt5$	116.565	180.0
g-h-i-p-t	$\left[\begin{array}{ccc}2\sin4\theta_p & 2\cos4\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	108.0
i-j-a-q-p	$\left[\begin{array}{ccc}2\sin3\theta_p & 2\cos3\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	36.0
a-b-c-r-q	$\left[\begin{array}{ccc}2\sin2\theta_p & 2\cos2\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	-36.0
c-d-e-s-r	$\left[\begin{array}{ccc}2\sin\theta_p & 2\cos\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	-108.0
p-q-r-s-t (bottom)	$\left[\begin{array}{ccc}0 & 0 & -1\end{array}\right]^T$	180	Undefined

As a second example, and one for which sundials will be designed, consider a dodecahedron with an orientation that results from a 180° rotation about the z-axis of the one in Figure 5. Unit normal-vectors of the facets in this new orientation are obtained by operating on each vector of Table 2 with the rotation matrix $R_z(\varphi_z)$ with $\varphi_z = \pi$, so $R_z(\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

This transforms any unit normal-vector $\vec{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$ into $R_z(\pi)\vec{u} = \begin{bmatrix} -u_x & -u_y & u_z \end{bmatrix}^T$, yielding the unit vectors in Table 3.

Table 3. Dodecahedron parameters when rotated $180°$ about the z-axis

dodecahedron facet (Figure 5 rotated 180°)	unit normal vector $(\theta_p = 72°)$	inclination angle (degrees)	declination angle (degrees)
k-l-m-n-o (top)	$\left[\begin{array}{ccc}0 & 0 & 1\end{array}\right]^T$	0	Undefined
a-b-l-k-j	$\left[\begin{array}{ccc}0 & 2 & 1\end{array}\right]^T/\sqrt5$	63.435	180
b-c-d-m-l	$\left[\begin{array}{ccc}-2\sin\theta_p & 2\cos\theta_p & 1\end{array}\right]^T/\sqrt5$	63.435	108
d-e-f-n-m	$\left[\begin{array}{ccc}-2\sin2\theta_p & 2\cos2\theta_p & 1\end{array}\right]^T/\sqrt5$	63.435	36
f-g-h-o-n	$\left[\begin{array}{ccc}-2\sin3\theta_p & 2\cos3\theta_p & 1\end{array}\right]^T/\sqrt5$	63.435	-36
h-i-j-k-o	$\left[\begin{array}{ccc}-2\sin4\theta_p & 2\cos4\theta_p & 1\end{array}\right]^T/\sqrt5$	63.435	-108
e-f-g-t-s	$\left[\begin{array}{ccc}0 & -2 & -1\end{array}\right]^T/\sqrt5$	116.565	0
g-h-i-p-t	$\left[\begin{array}{ccc}-2\sin4\theta_p & -2\cos4\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	-72
i-j-a-q-p	$\left[\begin{array}{ccc}-2\sin3\theta_p & -2\cos3\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	-144
a-b-c-r-q	$\left[\begin{array}{ccc}-2\sin2\theta_p & -2\cos2\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	144
c-d-e-s-r	$\left[\begin{array}{ccc}-2\sin\theta_p & -2\cos\theta_p & -1\end{array}\right]^T/\sqrt5$	116.565	72
p-q-r-s-t (bottom)	$\left[\begin{array}{ccc}0 & 0 & -1\end{array}\right]^T$	180	Undefined

The initial step will be to use the *Sonne* software to design the sundials for the facets selected to contain a sundial, with each sundial separately designed. This software requires as user-supplied data: a choice of sundial type; the latitude and longitude of the location where the sundial will be placed; the inclination and declination angles of the facet to have the sundial; and, the size of the sundial and gnomon height. The sundial type chosen for all facets is *inclining/reclining*. The location used for the design is St Louis, MO, at $38.654°N$, $90.324°W$. The assumed orientation of the dodecagon is that of Table 3, where the inclination and declination angles are listed. Program default values for size are used for the sundial plate and gnomon (rectangular plate $Height \times Width = 190 \times 160$, gnomon height 30 mm). The base length of the gnomon is selected to be 0 mm for a pole gnomon. Also selected are dates for displaying declination lines; these include the solstice and equinox dates and, for some dials, family birthdates. With these parameters, *Sonne* produces an on-screen drawing of the sundial, showing its hour lines, pole gnomon location, and declination lines. There is also a corner scale indicating the gnomon height and providing a convenient drawing-scale reference. The next step is to place the *Sonne* drawing onto the pentagon-shaped facet for which it was designed. This is accomplished by doing a screen capture of the *Sonne* drawing and pasting that into a blank page of the 2D drafting software *Visio*. This pasted drawing is then scaled to achieve a gnomon height of 25.4 mm (1 in.) A drawing of the regular pentagon is then superimposed onto the same page, with the size of the pentagon selected as desired, as in Figure 8 this selection is made so the circumscribed circle touching the five vertices has a radius of 2.5 in. The hour and declination lines are then traced into *Visio* vector format and trimmed to fit within the pentagon. Text (such as hour numbers), dial furniture, and color choices are then added. An example is shown in Figure 9.
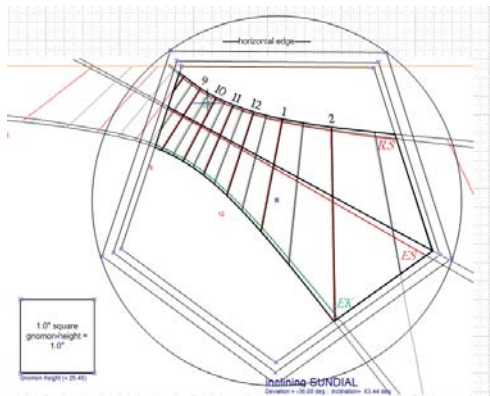

Figure 8. Visio page showing *Sonne* image, pentagon diagram, and traced hour and declination lines
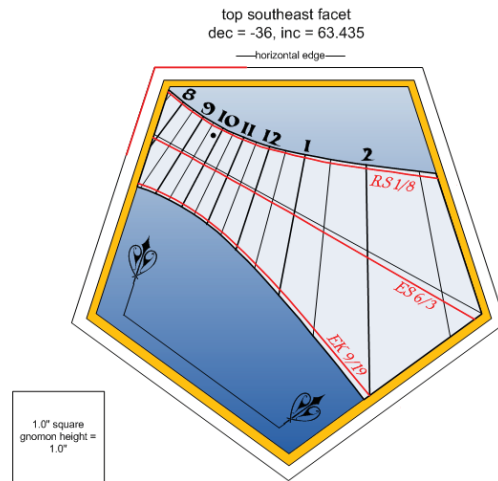

Figure 9. Southeast facet

Method of Construction

The polyhedral sundial shown in Figures 15-16 is made of wood with the dial plates printed on paper that is glued to the facets. The wood selected is African mahogany, milled to a thickness of 0.5 in. and cut into twelve 5.5-by-5.5 in. square blanks. Each blank is to be cut into a regular pentagon sized so that each vertex touches a circumscribed circle of radius 2.5 in. Each edge of a pentagon is to be beveled at an angle of $58.2825°$, which is one-half the dihedral angle of the facets of a dodecahedron. These cuts are accomplished using the method of pattern sawing on a table saw; see S. Latta [17] for a discussion of this method. The pattern used is a regular pentagon having a circumscribed circle of radius

$$2.5 - 0.5\left[\cos\left(36°\right)\tan\left(58.2825°\right)\right]^{-1} \approx 2.118 \text{ in. and no bevel; the pattern is diagrammed in Figure 10 as}$$

the dashed-line that is concentric with the pentagonal facet (shown as a solid line) of the dodecahedron. A pattern of 1/8 in. plastic is made inexpensively with a laser-cutting CNC machine by an engraving

company. The plastic pattern is replicated in MDF of thickness 5/8 in. using a bandsaw and spindle sander. Accuracy to achieve a good approximation to regular pentagon is important in making the MDF pattern. The blade of the table saw is tilted to achieve the desired bevel angle to a good approximation. The setup for using the pattern to cut the pentagon facets is seen in Figure 11. The pattern is attached to a mahogany blank by using two-sided carpet tape. It is important that the pattern remain in a fixed position on the blank throughout the cutting process. To help insure this, three screws are placed in the pattern with their tips slightly protruding from the side attached to the blank. These and the carpet tape result in an adequately rigid connection. The result of cutting one side of the blank is shown in Figure 12. Five cuts using the pattern and pattern guide in this way completes making one of the twelve pentagonal facets needed to construct the dodecahedron.
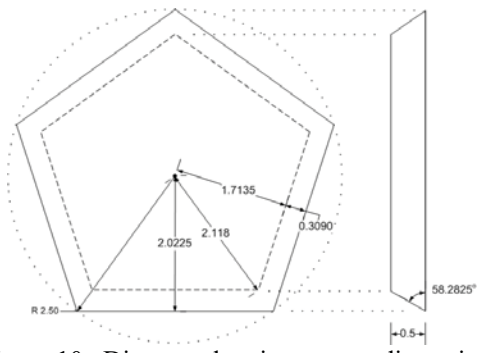

Figure 10. Diagram showing pattern dimensions
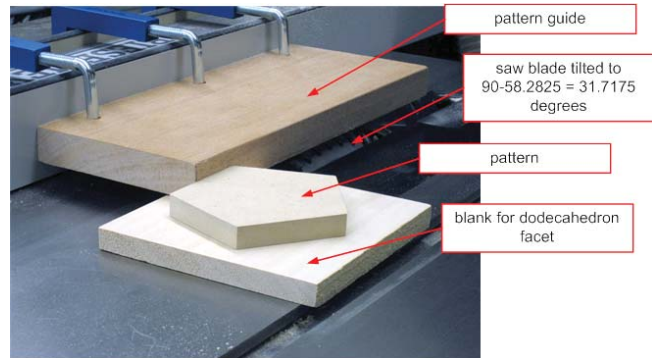

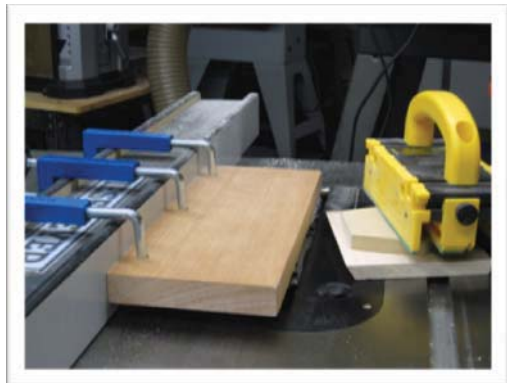Figure 11. Table saw setup for cutting pentagonal facets using a pattern

Figure 12. Result of one cutting pass using the pattern

Figure 13. Dodecahedron after glue up and sanding

Once they are cut, eleven of the facets are glued together. The remaining facet serves as the bottom and is retained in a manner that permits it to be removed for accessing the interior of the dodecahedron should that become necessary at a later time. The resulting dodecahedron is shown in Figure 13. The polyhedral sundial is completed by gluing sundial plates printed on paper to the facets and then mounting the pole gnomons. The gnomons, shown in Figure 14, are made of brass. The center of the spherical nodus at the top of every pole is 1 in. above the dial plate.
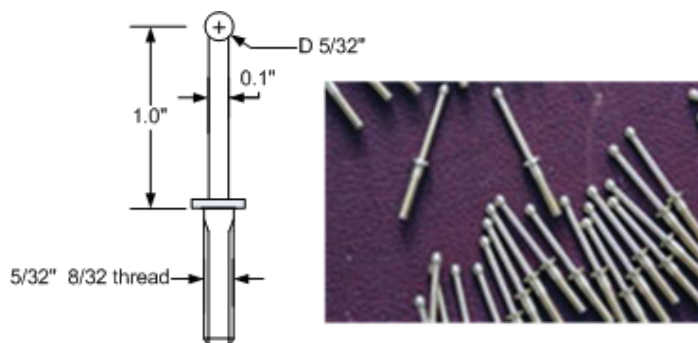

Figure 14. Gnomon design

They were custom made by machinist William Turner of Turner & Associates in St. Louis. Views of the completed sundial are displayed in Figures 15 and 16.

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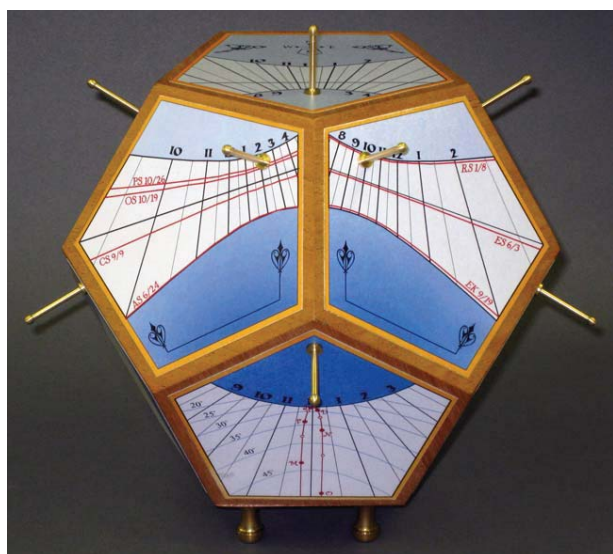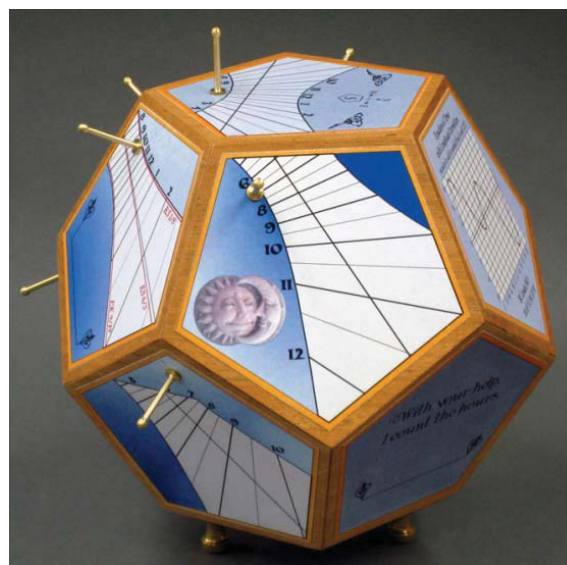
Figure 15. South facing facets

Figure 16. East facing facets

Donald L. Snyder, 7385 Pershing Ave., Unit #2, St. Louis MO 63130   dls@ee.wustl.edu