Solar Shadows and Analemmatic Sundials

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Abstract

Methods for predicting the geometry of shadows cast by sunlit objects are well known. Some of these are reviewed and then applied to the design of a particular kind of interactive solar clock known as an analemmatic sundial.

Contents

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I. Introduction

Shadows cast by objects exposed to sunlight change with time and date as the sun's apparent position varies during the day due to the earth's rotation and during the seasons due to the earth's orbital motion around the sun. Understanding how to predict the behavior of solar shadows is important in a number of areas, including architectural design and the placement of building structures, the design and placement of solar-energy collectors, the design and placement of flower and vegetable gardens, and the design and placement of time and date indicators.

This discussion will be about solar shadows with application to sundials. At first thought, it may seem that sundials for indicating time are of little practical value in these days of high technology. After all, modern clocks are much more accurate than sundials, they work at night as well as day, they work on rainy days as well as sunny ones, and they are cheap and readily available. The study of sundials is worthwhile nonetheless. Understanding how sundials work helps to understand the heating and cooling requirements within buildings exposed to sunlight, the energy produced by solar panels, the growth of flowers and vegetables in gardens, and effects in other applications where sunlight plays a role. Moreover, shadows cast by sundials are useful in unexpected, modern ways.   An example is the sundial affixed to the roving vehicle that landed on Mars in 2004, shown in Figure 1.[2] Also, and importantly, the study of sundials has educational merit. It helps students learn about the world around them, and it shows them how some of the abstract mathematics they learn in school, particularly trigonometry, provide very practical tools. Moreover, designing and making one's own personal sundials can be fun!

The first topic to be studied is solar shadows, identifying where the sun is relative to any place of interest and where the shadows of objects fall that result from its light. The ideas are then used for the design of analemmatic sundials. These are interactive sundials, and many examples of them exist.

Credit: NASA/JPL/Cornell   Credit: NASA/JPL/Cornell   Credit: NASA/JPL/Cornell
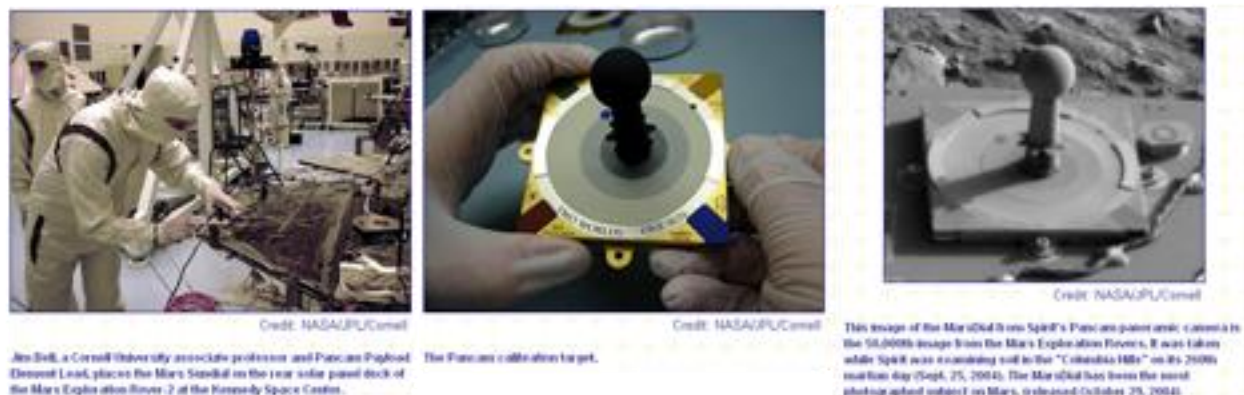
Figure 1 (left) The sundial is installed on the deck of the solar array on the Mars Rover. (center) The sundial on the Mars Rover. (right) A picture taken of the sundial while on Mars, showing the shadow.

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[2] These images are from the following website at Cornell University http://athena.cornell.edu/kids/sundial.html. More can be learned about the dial from the website of the Planetary Society http://www.planetary.org/rrgtm/marsdial/ and other sites on the internet.

Examples of Analemmatic Sundials

The shadow casting object in an analemmatic sundial, called the *gnomon*, is usually a vertical rod or a person. In contrast to most sundials, the gnomon is not in a single, fixed position but, rather, must be placed in a varying position that depends on the date in order for the sundial to indicate the correct time for that date. A person serving as the gnomon must stand on a date-dependent place to read the time with any accuracy. Analemmatic sundials have been built at many locations around the world and in many different styles. Some examples are in the pictures shown in Figure 2; these pictures were derived from a search of the internet for "analemmatic sundials." The bottom picture on the left is located at the Brooklyn Children's Museum. It was made by Robert Adzema using hundreds of one-inch tiles. The gnomon is not always a person, as seen in the table top dial of Figure 3, where a thin rod is used as the shadow caster. This dial was made by John Carmichael (see http://www.sundialsculptures.com/).

It can be seen in these pictures that an analemmatic sundial consists of two main parts. One part has some marks along a curve that indicate the hours of time; this curve is an ellipse, as discussed below. The other part of the dial is a platform containing the date marks where the person or gnomon stands to cast the shadow towards the hour marks. The design of an analemmatic sundial therefore consists of locating the hour marks along the ellipse and the date marks on the platform. Once these numerical aspects of the design are completed, there is then much leeway in completing the artistic features of the design to give the dial its unique style and character. A description of solar shadows is needed to locate the hour and date marks. This is developed in the next section.

There are various approaches for designing an analemmatic sundial. Purely graphical methods can be used; see, for example, Rohr [4]. Empirical methods can also be used by making daily and yearly observations of the shadow of a vertical rod. An ellipse is first laid out at the intended site of the sundial, with the minor axis oriented in the north-south direction. Time marks can be placed on the ellipse at selected times during a day of choice, such as hourly, by noting where the shadow of a vertical rod located on the minor axis intersects the ellipse. A date mark can be placed on the minor axis of the ellipse for that day. Observations over the course of a year will be needed to locate other date marks empirically along the minor axis. This experimental method of construction is protracted over time and tedious. Fortunately, there is a very nice analytical way of doing it, which is developed as our discussion proceeds. While this analytic method for designing analemmatic sundials can also be used for designing other types of sundials and solar calendars, including sundials with fixed shadow casters and henges, we will not explore these extensions here.

II. Solar Shadows

To a very good approximation, the earth rotates around the sun in an elliptical orbit. The orbit is not quite an ellipse because of the gravitational pull of other planets and distant stars, but the deviation from an ellipse is so small that it can be disregarded for the purpose of designing solar clocks.

Figure 2 Pictures of analemmatic sundials obtained
by searching the internet for sundials of this type

Figure 3 Table top analemmatic sundial made by John Carmichael

A. A Brief Review of Ellipses

A standard ellipse is a curve in a plane, as shown in Figure 4. It has major and minor axes that can be aligned with the axes of a Cartesian coordinate system, with its center, C,
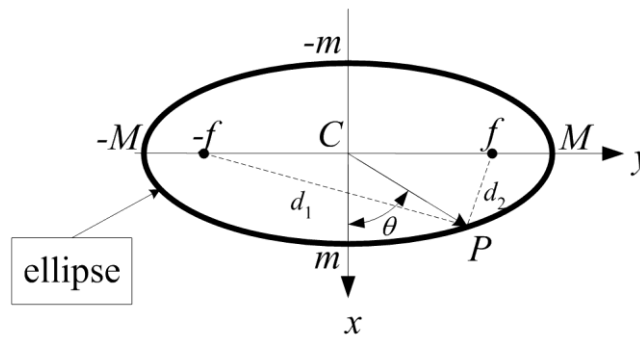

Figure 4 A standard ellipse

located at the origin (0,0) of the coordinate system and its short and long axes aligned with the x-axis and y-axis, respectively, of the coordinate system.[3] The equation for points (x, y) that lie on the elliptical curve is

$$\frac{x^2}{m^2} + \frac{y^2}{M^2} = 1 \tag{1}$$

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[3] The orientation of the coordinate system used here is rotated clockwise by 90° from ordinary usage. It is a "right-handed" system with positive angles measured counterclockwise from the downward pointing positive x-axis. A z-axis in a three-dimensional system would be added to point vertically up out of the x,y-plane. The motivation for having the positive x-axis downward pointing is that this will be towards south in sundial designs. A compass with the same coordinate system will have north-south and east-west directions in their usual orientation with north up. Sun azimuth angles are usually measured from south in sundial design, so we select the positive x direction down, which is towards south on the compass.

Where M and m are parameters that determine the size of the major and minor axes. If M and m are equal, say to r, then the ellipse is a circle of radius r. If $M > m$, then the major $axis$ of the ellipse lies along the y axis of the coordinate system, as shown in Figure 4. Another way to think of the ellipse is in terms of the angle θ shown in the figure. A point P located in the plane at

$$(x, y) = (m\cos\theta, M\sin\theta) \qquad (2)$$

is on the ellipse because

$$\frac{(m\cos\theta)^2}{m^2} + \frac{(M\sin\theta)^2}{M^2} = 1.$$

As θ varies from 0° to 360°, the point P moves from $(x, y) = (m, 0)$ counterclockwise around the ellipse and back. The center C of the standard ellipse lies at the origin $(0,0)$ of the coordinate system. A point P on the ellipse is at a distance $\sqrt{(m\cos\theta)^2 + (M\sin\theta)^2}$ from the center C of the ellipse; this distance varies with θ and, hence, the location of P on the ellipse. The points $(0,-f)$ and $(0,+f)$ are called the *foci* of the ellipse if $f = \sqrt{M^2 - m^2}$. The distance d_1 from the focal point at $(0,-f)$ to a point P at $(m\cos\theta, M\sin\theta)$ on the ellipse is

$$
\begin{aligned}
d_1 &= \sqrt{(m\cos\theta)^2 + (M\sin\theta)^2} \\
&= \sqrt{M^2 - m^2 + 2Mf\sin\theta + M^2\sin^2\theta + m^2(1 - \sin^2\theta)} \\
&= \sqrt{M^2 2Mf\sin\theta + f^2\sin^2\theta} \\
&= M + f\sin\theta.
\end{aligned} \qquad (3)
$$

Similarly, the distance d_2 from the focal point at $(0, f)$ to P is $M - f\sin\theta$. This yields the important property that for any point P on the ellipse, the sum of the distances from the two foci to that point equals the length, $2M$, of the major axis,

$$d_1 + d_2 = 2M. \qquad (4)$$

The *eccentricity*, e, of an ellipse is defined by

$$e = \sqrt{1 - \frac{m^2}{M^2}} = \frac{f}{M} \qquad (5)$$

The eccentricity has a value between 0 and 1, $0 \le e \le 1$. An ellipse having an eccentricity e = 0 is a circle. As illustrated in Figure 5, the ellipse departs more and more from a circle as e increases, becoming simply a line when $e = 1$.
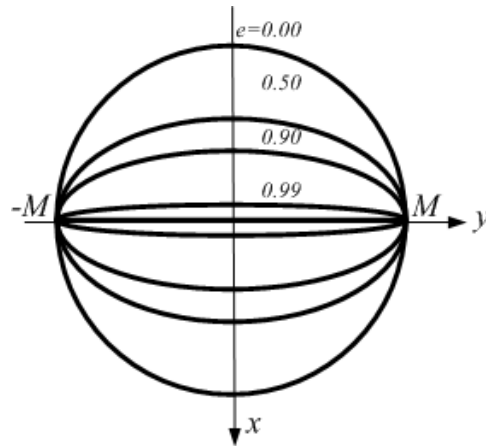
Figure 5 Illustration of the effect of eccentricity on the shape of an ellipse

The equation for an ellipse that is not centered at the origin (0,0) but rather at a point (x_0, y_0) is

$$\frac{(x-x_0)^2}{m^2} + \frac{(y-y_0)^2}{M^2} = 1 \tag{6}$$

An ellipse centered at $(0,-f)$ is shown in Figure 6. The focal point at $(0, f)$ in Figure 4 is now at the origin in Figure 6, and (6) becomes

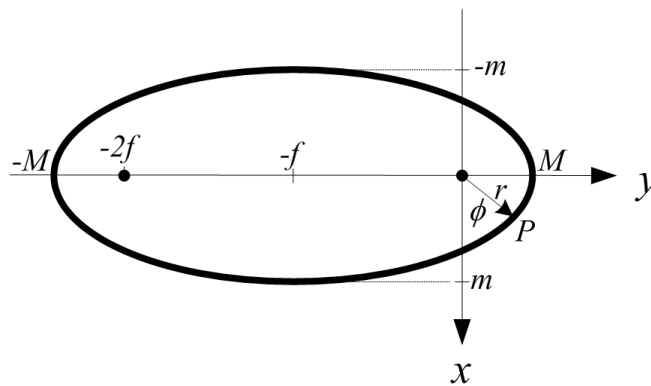$$\frac{x^2}{m^2} + \frac{(y+f)^2}{M^2} = 1 \tag{7}$$

Figure 6 Ellipse with a focal point at the origin

A point P on this ellipse has coordinates $(x, y) = (r\cos\phi, r\sin\phi)$, where r is the distance from the focal point at the origin to P, and ϕ is the angle that the line connecting the origin to P

makes with the positive x axis. The point P at $(r,\phi)=(M-f,+90°)$ on the ellipse is closest to the focal point at the origin, and $(r,\phi)=(2M-f,-90°)$ is the most distant point.

B. Earth Motions and Seasons

The earth orbits the sun periodically, with a period of 365.25 days, following an elliptical path with the sun at one focal point, so we may think of the point P in Figure 6 as representing the center of the earth and the origin as representing the center of the sun. The orbital path defines a plane called the *ecliptic*. The center of the earth, throughout the course of its annual orbit, and the center of the sun lie in the ecliptic. While the earth flies through space around the sun, it also rotates periodically about its own axis, with a period of 24 hours.

The closest approach of the earth to the sun, called the *perihelion*, occurs around January 2 each year. At perihelion, $\phi=90°$ in Figure 6, and r is approximately 1.47×10^{11} meters (or about 91 million miles). The point when the earth is most distant from the sun, called *aphelion*, occurs around July 3. At aphelion, $\phi=-90°$ in Figure 6, and r is approximately 1.52×10^{11} meters (or about 94 million miles). The eccentricity e of the earth's orbit is approximately 0.0167, so the orbit is very nearly a circle. This eccentricity is too small to account for the annual seasons. The earth rotates about its polar axis as it orbits the sun. This axis is not perpendicular to the ecliptic but, rather, tilts about 23.45 degrees from that. The tilt of the axis, along with the orbital motion, cause the angle of incidence of the sun's rays at any place on the earth to vary between $-23.45°$ and $+23.45°$, resulting in the seasons. Figure 7 shows some of the critical positions of the earth in its elliptical orbit around the sun. Consider a plane that is perpendicular to the ecliptic and which contains the centers of
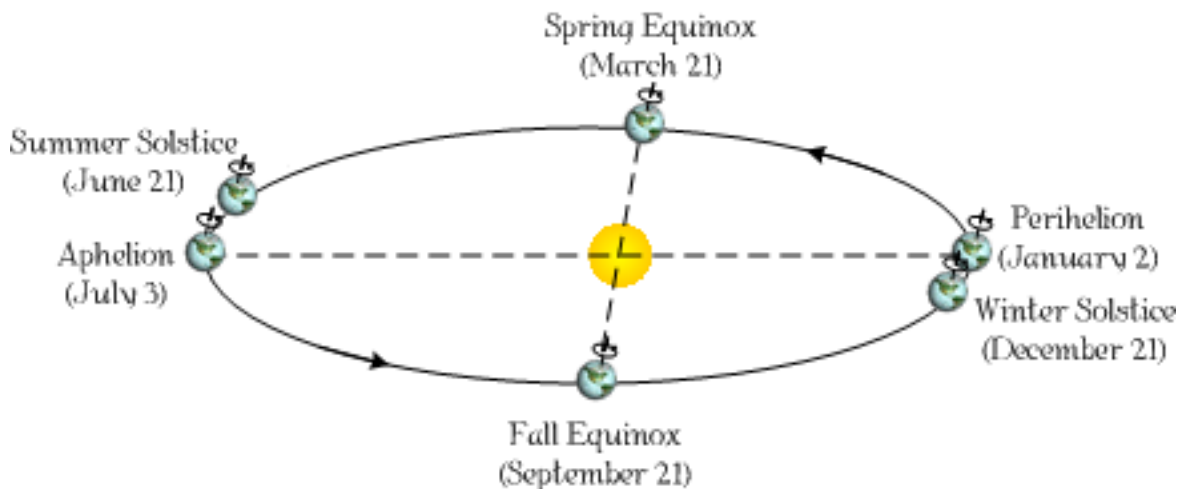

Figure 7 Some critical positions of the earth in its elliptical orbit around the sun

the sun and earth. I will call this the *Milankovitch plane* after the Serbian scientist Milankovitch who in the 1920s studied the influence of the solar cycle on climate (see

http://aa.usno.navy.mil/faq/docs/seasons_orbit.html).　　　Over the course of a year, the Milankovitch plane rotates around the sun, moving with the earth in its orbit. The axis of rotation of the earth lies in this plane only two times during the year. These are the winter and summer solstices.   At a solstice, the greatest number of hours of sunshine for any day of the year occurs in the hemisphere tilted towards the sun, and this is in the summer for that hemisphere; the fewest number of hours occurs for the opposite hemisphere, which is in the winter. The winter and summer solstices indicated in Figure 7 are for the northern hemisphere. At other days of the year, the axis of rotation of the earth is tilted out of the Milankovitch plane. The greatest tilt occurs twice a year when it reaches ±23.45 degrees. These are the equinoctial days when the number of hours in the day and night are equal.

The orbits of all of the other planets of our solar system also lie in the ecliptic plane. Shown in Figure 8 is a photograph taken shortly after sunset by Jimmy Westlake on June 19, 2005. Three planets can be seen above the Colorado Rocky Mountains in the foreground. These are Saturn, Venus, and Mercury, all lying in the ecliptic. Also, see Figure 16 for an annotated version of this picture.

Figure 8 Photograph taken by Jimmy Westlake of Colorado Mountain College on June 19, 2005 shortly after sunset. Saturn, Venus, and Mercury can be seen to lie along a line, the ecliptic, with the Colorado Rocky Mountain skyline in the foreground. (From the website of the Astronomy Picture of the Day for June 24, 2005 at http://antwrp.gsfc.nasa.gov/apod/archivepix.html)

C. Coordinates

The earth, approximated as a sphere, is represented in Figure 9. The two points where the spin axis of the earth meets the surface of the earth are called the *poles*. One end of the spin axis points towards the star Polaris. The pole closest to Polaris is called the *North Pole*, and the other is the *South Pole*. The *equator* is the great circle[4] that is perpendicular to the spin axis and mid-way between the poles.   Great circles that pass through the poles are called *meridians*. By a long standing tradition, the meridian passing through Greenwich, England is known as the *prime meridian*; it is used as a reference to specify the position of all other meridians. Circles that are not great circles but are parallel to the equator are called *parallels*.

Identifying the location of an object on the earth's surface or in space requires that a co-ordinate system be specified. There are many possible coordinate systems that can be adopted. Some are earth centered, and some are place centered.

1. Earth Centered Coordinate Systems

The *equatorial coordinate system* is earth centered. It is defined in terms of the *equatorial plane*. This is a plane of infinite extent that passes through the center of the earth and contains the earth's equator. In the equatorial coordinate system, the location of any point in space can be specified by the values of its Cartesian coordinates (x, y, z) shown in Figure 9.
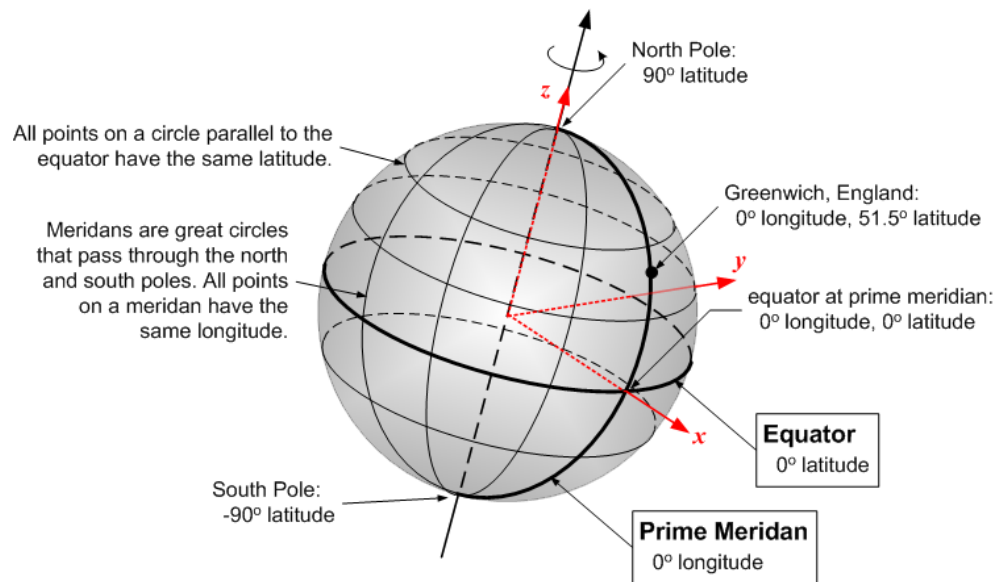

Figure 9 The equatorial coordinate system

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[4] A *great circle* on the surface of a sphere is a circle on the surface with its center at the center of the sphere.

The origin of this coordinate system is at the center of the earth. It is a right-handed coordinate system, with the x axis oriented towards the intersection of the prime meridian and the equator, and the z axis is oriented towards the North Pole. A point in the equatorial plane has Cartesian coordinates $(x, y, 0)$. A point P on the surface of the earth will have coordinates (x, y, z) that satisfy $x^2 + y^2 + z^2 = r^2$, where r is the earth's (mean) radius. Alternatively, and commonly, the location of the point can be specified in a three-dimensional polar coordinate system, having two angular coordinates and one radial coordinate. These are defined in Figure 10. The *longitude* of P is the angle θ measured along the equator between the prime meridian and the "local" meridian passing through P. This angle is positive if measured counterclockwise from the x axis in the $x,y-$ plane (that is, towards the east from the prime meridian in the equatorial plane). Otherwise, it is negative. for example, the longitude of St. Louis, MO, is 90.3 degrees west of the prime meridian, so $\theta = -90.3°$ or, alternatively, $\theta = 269.7°$. The *latitude* of P is the angle ϕ in Figure 10 measured along the
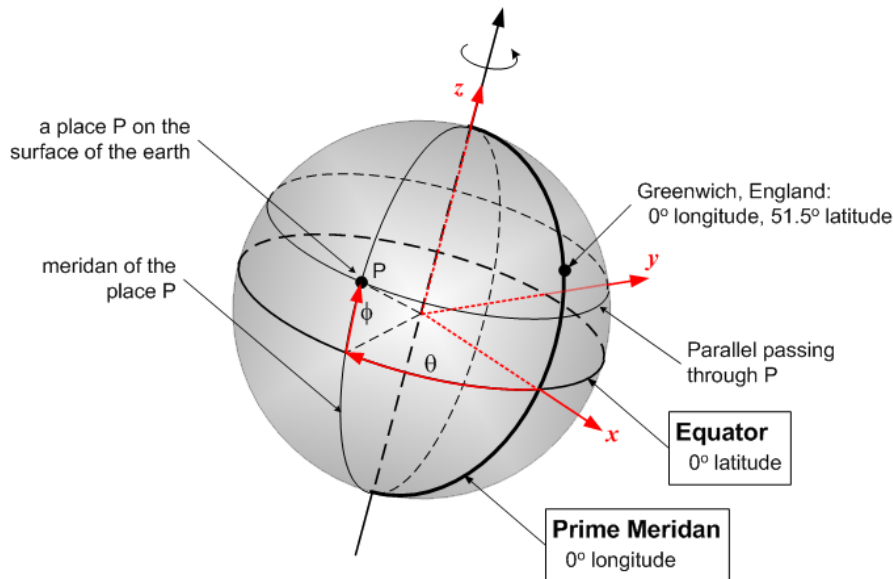

Figure 10 Longitude and latitude coordinates

local meridian that passes through P and the equator. This angle is the same as that between the equator and the point at the intersection of the parallel containing P with the prime meridian when measured along the prime meridian. It is positive if measured in the counterclockwise in the $x,z-$ plane (that is, towards the northern hemisphere from the x-axis). It is therefore positive for locations in the northern hemisphere and negative for those in the southern hemisphere.

The location of a place P on the surface of the earth can be specified by its latitude and longitude angles, ϕ and θ, and its distance, r, from the earth's center. The distance from earth's center is usually omitted explicitly when a spherical earth model is assumed, but the elevation above or below sea level is given when deviations from a sphere are of interest. For sundial computations, this small variation is usually disregarded. Alternatively, the location of P can be given in terms of its (x, y, z) coordinates. These are related to the polar-coordinate representation in the following way:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{bmatrix} \qquad (8)$$

.

Finally, note that the location of any object, whether on the surface of the earth or not, can be specified within this coordinate system. All that is required is to consider a line drawn from the center of the earth to the object's center. The point where this line penetrates the surface of the earth specifies the latitude and longitude of the object, and its radial distance completes the specification. This includes the sun, an object in which we are very interested.

Another earth-centered coordinate system is one in which the longitudinal reference is not the prime meridian but, rather, the local meridian of a place P of interest. This system is illustrated in Figure 11 for specifying the position of the sun. The Cartesian coordinate system (x', y', z') is obtained from that of Figure 10 by rotating the coordinates
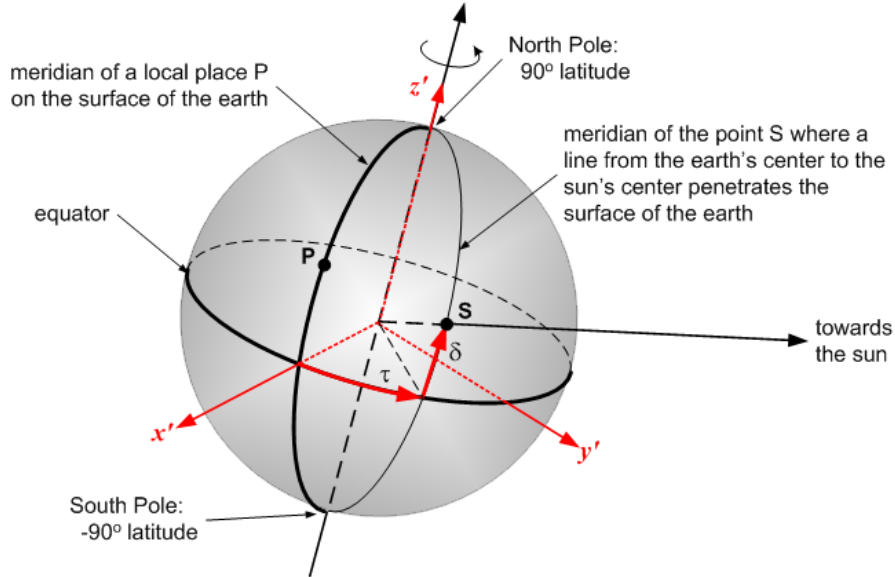
Figure 11 Hour line and declination coordinates

(x, y, z) about the z axis through an angle θ equal to the longitude of P. A point located at the coordinates (x, y, z) in Figure 9 is located at the coordinates (x', y', z') in Figure 10, according to:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{bmatrix}. \qquad (9)$$

A polar coordinate representation in the (x', y', z') frame is important and used routinely in navigation, sundial design, and other areas involving solar effects. There are two angles, which are analogous to the longitude and latitude angles of Figure 10. The longitude angle relative to the local meridian of P is labeled τ in Figure 11. It is common to specify this angle in the units of time rather than degrees. This is achieved taking into account that the earth rotates through 360 degrees in each 24 hour day, so it rotates through 15 degrees per hour and 1/4 degree each minute. This scaling is used in specifying τ in time units. The sun passes through the plane of the local meridian of P at 12:00 noon local solar time. At that same instant, it passes through 11:00 a.m. and 1:00 p.m. at the meridians that differ in longitude by +15 and -15 degrees, respectively, from the longitude of P. The value of $\tau = 1.5$ hours represents the meridian separated towards the east by 22.5 degrees of longitude from the local meridian of P. Noon occurs at that meridian 1.5 hours before it does at P. Similarly, $\tau = -1.5$ marks the meridian 22.5 degrees of longitude towards the west from that of P, and noon occurs there 1.5 hours after it does at P. In general, $\tau = -15\left(t_{solar-time} - 12\right)$ degrees. The latitude angle of the point S in Figure 11 is labeled δ. This latitude angle, measured from the equatorial plane, is called the declination when the point S is determined by the position of the sun, as it is in Figure 11. In this earth-centered coordinate system, the position of the sun is specified by its hour angle, τ, its declination, δ, and the distance, d_{sun}, of its center from the center of the earth. The sun's declination varies between $-23.45°$ and $+23.45°$ over the course of the sun in its annual orbit. The location the sun is given in terms of its (x', y', z') coordinates or its polar coordinates (τ, δ, r). These are related in the following way.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = d_{sun} \begin{bmatrix} \cos\delta\cos\tau \\ \cos\delta\sin\tau \\ \sin\delta \end{bmatrix}. \tag{10}$$

2. Place Centered Coordinate Systems

Coordinate systems that are centered at a place of interest on the surface of the earth, P, are also commonly used. An example is shown in Figure 12. Here, (x'', y'') lie in the plane that
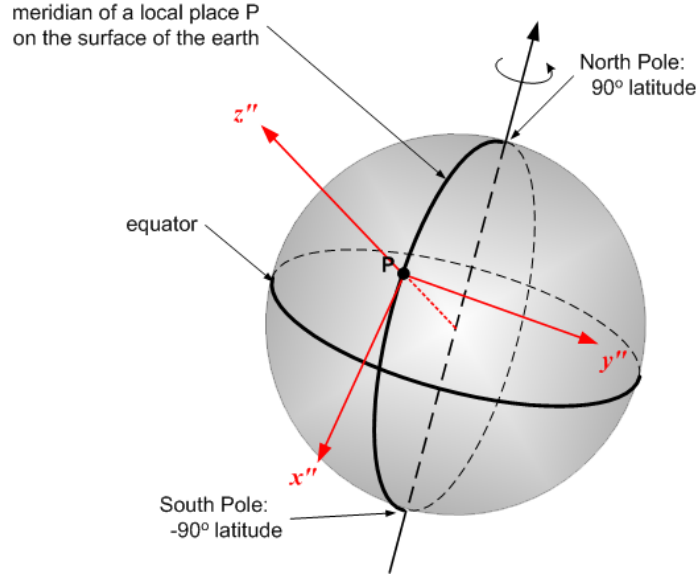
Figure 12 Horizon coordinates

is tangent to the earth's surface at P, which is called the *horizon plane*, with x'' lying in the meridian plane of P. The z'' coordinate is perpendicular to this plane at P. The coordinate system (x'', y'', z'') of Figure 12 is obtained from the coordinate system (x', y', z') of Figure 11 by translating the origin from the center of the earth to the place P on the surface and a rotation about the y' axis by $\tilde{\phi}$ degrees, where $\tilde{\phi}$ is the colatitude of P, defined by $\tilde{\phi} = 90° - \phi$. Thus,

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} + r_{earth} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\tilde{\phi} & 0 & -\sin\tilde{\phi} \\ 0 & 1 & 0 \\ \sin\tilde{\phi} & 0 & \cos\tilde{\phi} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \sin\phi & 0 & -\cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \qquad (11)$$

Where r_{earth} is the earth's radius. Hence,

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} x'\sin\phi - z'\cos\phi \\ y' \\ x'\cos\phi + z'\sin\phi \end{bmatrix} - r_{earth} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \qquad (12)$$

Since x'' lies in the meridian plane passing through the place P, it points towards the south; y'' points towards the east; and, z'' points towards the zenith at P.

A place-centered polar coordinate-system is natural and of interest for specifying the location of objects in space, such as the sun. This is the polar coordinate system shown in Figure 13. The Cartesian coordinates (x'', y'', z'') are those shown in Figure 12. The location
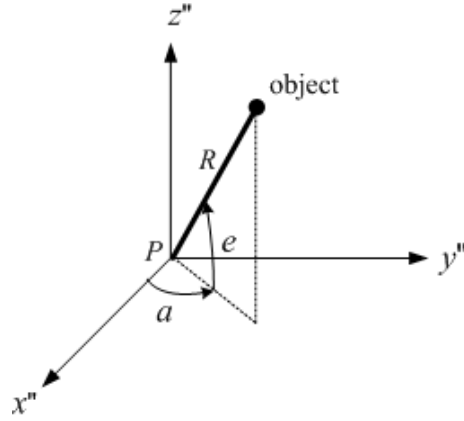
Figure 13 Polar horizon coordinates

of an object, such as the sun, is specified by two angles, a and e, and its radial distance R from the local point P on the earth's surface. Here, a is the *azimuth angle* of the object measured from the x'' axis, and e is the *elevation angle* of the object from the horizon plane. Positive values of the azimuth angle are measured counterclockwise from the positive x'' axis; that is, positive from south towards the east. The Cartesian and polar representations are related by:

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = R \begin{bmatrix} \cos e \cos a \\ \cos e \sin a \\ \sin e \end{bmatrix}. \tag{13}$$

Suppose that the sun is the object in Figure 13, so $R = d_{sun}$. By equating (12) to (13) and replacing (x', y', z') by their values in (10), there results

$$d_{sun} \begin{bmatrix} \cos e \cos a \\ \cos e \sin a \\ \sin e \end{bmatrix} = d_{un} \begin{bmatrix} \cos \delta \cos \tau \sin \phi - \sin \delta \cos \phi \\ \cos \delta \sin \tau \\ \cos \delta \cos \tau \cos \phi + \sin \delta \sin \phi \end{bmatrix} - r_{earth} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{14}$$

The following three equations are then obtained:

$$\cos e \cos a = \cos \delta \cos \tau \sin \phi - \sin \delta \cos \phi, \tag{15}$$

$$\cos e \sin a = \cos \delta \sin \tau, \tag{16}$$

and

$$\sin e = \cos \delta \cos \tau \cos \phi + \sin \delta \sin \phi - \frac{r_{earth}}{d_{sun}} \approx \cos \delta \cos \tau \cos \phi + \sin \delta \sin \phi, \tag{17}$$

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last update: January 23, 2006

where the ratio $r_{earth} / d_{sun} \approx 0.00004$ is neglected in the last term of (17) because of its small size. These are well known equations for determining the position of the sun at a given place for a specified date and time. The place provides the latitude, ϕ; the date provides the sun's declination, δ, either by calculation from known formulas or from an ephemeris table; and the time provides the hour angle. Given these parameters, the sun's elevation and azimuth can be determined. Here is a summary for later reference.

From (17), the solar elevation is:

$$e = \arcsin\left(\cos\delta\cos\tau\cos\phi + \sin\delta\sin\phi\right). \tag{18}$$

This elevation angle is measured positive from the horizon plane towards the zenith.

From the ratio of equations (15) and (16), the azimuth angle is:

$$a = \arctan\left(\frac{\sin\tau}{\cos\tau\sin\phi - \tan\delta\cos\phi}\right). \tag{19}$$

This azimuth angle is measured positive counterclockwise from south. For azimuth angles measured positive clockwise from the north, use $180 - a$ degrees. Care is needed in using (19) to insure that the angle is in the correct quadrant.[5]

$$
\begin{aligned}
\delta &= \quad \text{solar declination (determined from the date)} \\
\phi &= \quad \text{latitude (determined from the place)} \\
\tau &= \quad \text{time (solar hour angle from local south)}
\end{aligned}
$$

The declination of the Sun is zero at the vernal equinox, about March 21, and the autumnal equinox, about September 21. It's maximum and minimum values are $+23.45°$ and $-23.45°$, respectively. In the Northern hemisphere, the maximum occurs on the Summer solstice, about June 21, and the minimum on the Winter solstice, about December 21. The declination varies approximately as a sinusoid over the course of a year. Thus, the solar declination can be determined approximately by using the expression

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[5] If it is available, use the function $\text{atan2}(y, x)$ rather than $\text{atan}(x/y)$ to evaluate $\tan^{-1}(x/y)$. This will resolve the quadrant ambiguity issue when evaluating the inverse tangent function. The preferred function is available, for example, in MATLAB.

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$$\delta \approx 23.45^\circ \sin\left(2\pi \frac{N-81}{365.25}\right), \tag{20}$$

Where N is the day number starting with $N = 1$ on January 1, $N = 2$ on January 2, $\cdots$, $N = 81$ on March 21, etc. The graph of this expression in Figure 14 shows, approximately, the variation of the solar declination during a year. Of course, the declination of the sun varies continuously as a function of the date and time and not simply the day number. J. Meeus [3, Ch. 25] gives an accurate expression for the continuously varying solar declination in terms of the date and time.
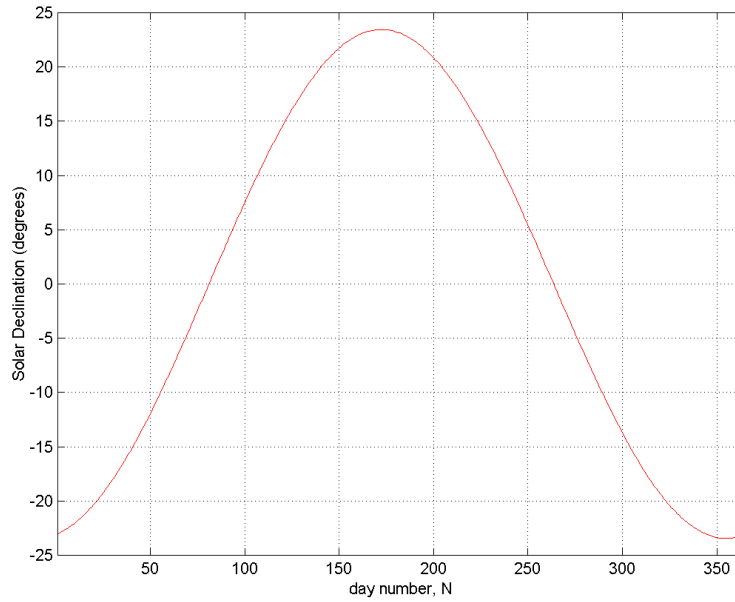

Figure 14 Solar Declination Versus Day Number

Example: Where is the sun at 12:00 noon (local solar time) on January 1 in St. Louis, MO? Where will the shadow be of a vertical rod that is 1 meter in length? The latitude of St. Louis, MO, is $\phi = 38.6^\circ$. To determine the solar declination, use the approximating formula (20) to get $\delta \approx 23.01$ degrees. The hour angle from local south is $\tau = -15(t_{solar-time} - 12) = 0$ degrees. Then, from (18), the elevation of the sun is 24.4 degrees, and from (19), its azimuth angle is 0 degrees (due south). The length and direction of the shadow cast by a thin, 1 m vertical rod is about 2.2 m towards true north. The length and direction at other times are given in the table below and illustrated in Figure 15.

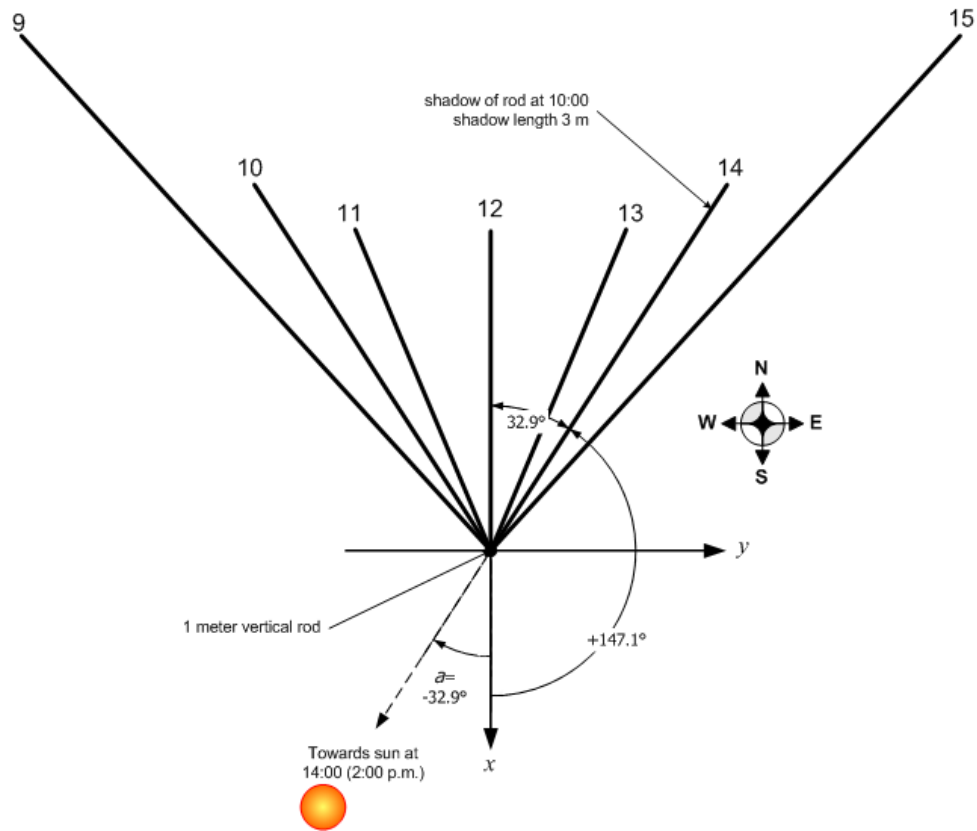| solar time	τ (degrees)	elevation angle above horizon e (degrees)	azimuth angle from south a (degrees)	shadow angle (degrees) $a+180°$	shadow length (m) ctan(e)
9:00	+45	11.7	+42.4	222.4	4.8
10:00	+30	18.5	+32.9	212.9	3.0
11:00	+15	22.9	+18.5	198.5	2.4
12:00	0	24.4	0	180	2.2
13:00	-15	22.9	-18.5	161.5	2.4
14:00	-30	18.5	-32.9	147.1	3.0
15:00	-45	11.7	-42.4	137.6	4.8

Figure 15 Shadow lines on January 1 in St. Louis, MO, of a one meter vertical rod

Example: Jimmy Westlake, who took the photograph displayed in Figure 8, teaches at Colorado Mountain College in Steamboat Springs, CO, which is at 40.5° north latitude. The angle which the horizon plane at a place of latitude ϕ makes with the equatorial plane is 90° – ϕ, so at Steamboat Springs, this angle is 90° – 40.5° = 49.5°. Since the equatorial plane is tilted at an angle of about 23.4° to the ecliptic plane, we expect the angle between the equatorial plane and the horizon plane at Steamboat Springs to be about 49.5° – 23.4° = 26.1° degrees. An estimate of this angle is seen in Figure 16.

III. Time

A. Varieties of Time

A quick answer to the question "What time is it?" is usually obtained by glancing at a wall clock or a wrist watch. But being quick does not necessarily mean being right. The complication is that there are many versions of time, so one should first inquire about the kind of time that is sought before answering. A 24 hour time scale starts at midnight with zero. Noon is at 12:00, and the sequential hours after noon are at 13:00, 14:00, etc., through 24:00, which is again midnight and so equal to zero in modulo-24 cyclic counting. The earth's elliptical orbit is often approximated as circular, with the earth rotating at a uniform rate of
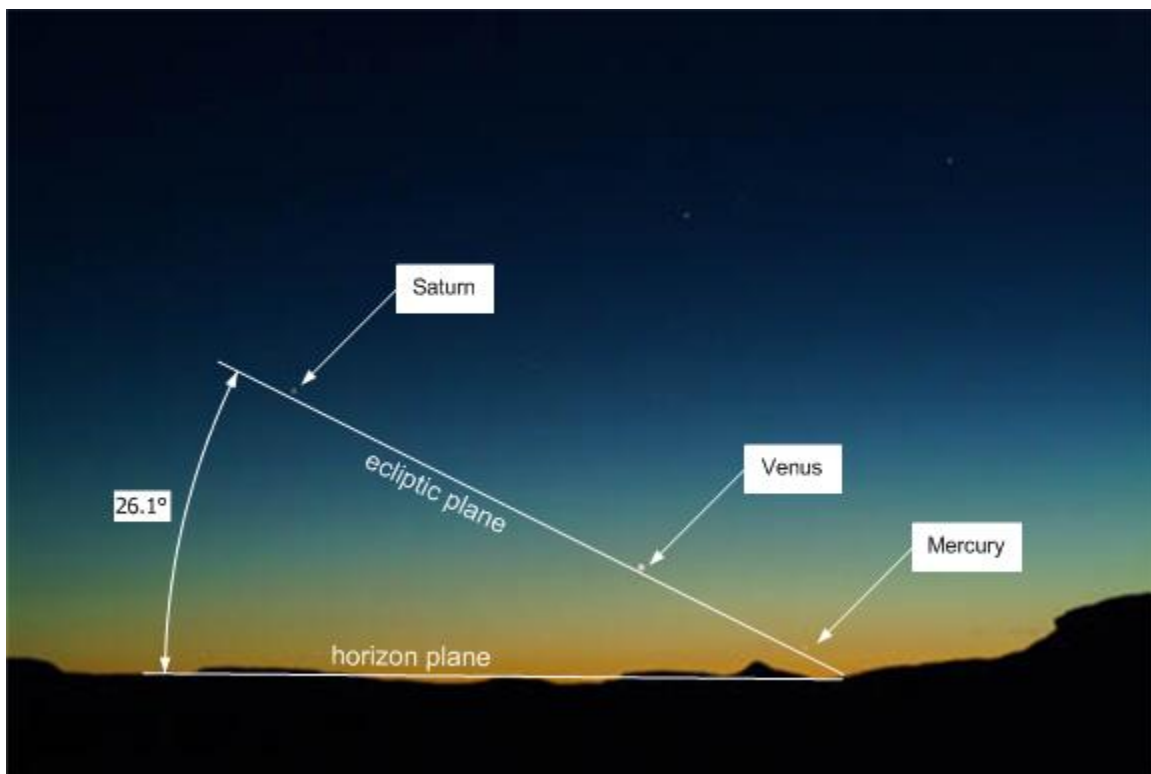

Figure 16 Annotated version of Fig. 6 showing Saturn, Venus, and Mercury in the ecliptic plane tilted about 26.1 degrees to the horizon plane

$360°$/year around the sun at the center of the circle. Solar time within this approximation is called *mean solar-time*. Sundials are routinely designed for mean solar-time. Solar time is then obtained from mean solar-time by a correction known as the *equation of time,* which is discussed below.

Local noon mean-solar-time, used in the discussion above, occurs when the sun's center lies in the plane of the local meridian, at which instant the sun is at its highest elevation that day for the local place. Local mean-solar-time is then measured by the sun's position relative to the

19

local meridian. 13:00 local mean-solar-time occurs one hour after the sun reaches its highest elevation at the location and the earth has rotated through 15 degrees. For most purposes, this is not a convenient way to measure time because it is different for every place having a different longitude. For this reason, other time measures have been adopted by governments so that a given time instant, such as noon, occurs for locations over a wide range of contiguous longitudes. Coordinated Universal Time (UTC), known also as Greenwich Mean Time (GMT) and Z-time, is perhaps the most important example. For this, the world is divided into time zones, with a time zone encompassing all locations within a span of 15 degrees of longitude. All locations within a zone have the same UTC, GMT or Z-time. The prime meridian, which is at the $0°$ longitude of Greenwich, England, serves as the reference in defining the zones. All meridians within ±7.5 degrees of the prime meridian form the central zone, labeled Z in the map of world time zones in Figure 17.[6] However, the zones are not a neat partitioning of the 360º degrees of longitudes into 15º groupings bounded by meridians. Rather, the zone boundaries are in many places very irregular, following rivers, state and political boundaries, etc. A common time is assigned to all locations within a zone regardless of its differing longitudes. Noon for all locations within a time zone occurs at the instant the sun's center is in the plane of the reference meridian assigned for that zone, with the reference meridian being at the center of the 15º span of meridians of the zone. The eastern time zone in the U. S. is in the zone labeled R in Figure 17. Times within the eastern time zone are labeled Eastern Standard Time (EST). EST equals GMT minus 5 hours because zone R is (about) 75º of longitude west of the prime meridian at Greenwich, England. There are also Central Standard Time (CST), Pacific Standard Time (PST), and other special designations around the world. There are thus many kinds of time measures, and this also includes daylight savings times that have not yet been mentioned. Relating local solar-time to local "clock" time requires accounting for the difference between the longitude of the local meridian and that of the reference meridian for the time zone in which the local place is situated, and whether or not daylight savings time is in effect must be considered. Thus,

$$t_{clock-time} = t_{solar-time} + \frac{1}{15}\Delta L° - DST = t_{mean-solar-time} - EOT + \frac{1}{15}\Delta L° - DST, \qquad (21)$$

where $\Delta L°$ is the longitude correction,

$$DST = \begin{cases} 1, & \text{daylight savings time in effect} \\ 0, & \text{daylight savings time not in effect.} \end{cases} \qquad (22)$$

and

$$EOT = t_{solar-time} - t_{mean-solar-time} \qquad (23)$$

is the "equation of time," which is discussed more below.

There is additional complication that makes answering the question "What time is it?" even more difficult. The measurement of time by fundamental physical processes has in recent

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[6] See http://www.maybeck.com/ztime/ for an interesting discussion by Harold Maybeck of the alphabetic labeling of time zones.

years resulted in the atomic clock. With it, time is measured by natural oscillations of atoms. Important devices rely on this method of measuring time, such as: global positioning systems now found in many cars, navigation, and military systems; watches and clocks automatically updated by time-synchronized radio transmissions; and, time on the internet. While a year measured by atomic time and solar time differ by about a "leap second" that might seem insignificant, the discrepancy is far from negligible. There is much discussion currently about how to modify universal coordinated time (UTC) to correct the problem. Arguments between engineers and scientists who rely on atomic time and astronomers and sundialists who rely on solar time are intense about this one second annual variance. See:

> http://www.ucolick.org/~sla/leapsecs/,
> http://www.cl.cam.ac.uk/~mgk25/time/metrologia-leapsecond.pdf ,

and

> http://www.ucolick.org/~sla/leapsecs/onlinebib.html

for a survey and references on leap seconds. See

> http://www.mail-archive.com/leapsecs@rom.usno.navy.mil/msg00476.html

for a discussion by J. Meeus of the impact on astronomy and sundials of switching UTC to an atomic time measure.

B. The Equation of Time

The *Equation of Time* (*EOT*) is the difference between solar time and mean solar time. This correction depends on the declination of the sun and so is date and time dependent. Mean solar time is determined under the approximation that the earth's orbit is circular and in the equatorial plane. Solar time and mean solar time differ because the earth's orbit is elliptical, not circular, and the ecliptic is tilted away from the equatorial plane. Robert Urschel gives an excellent tutorial discussion of the EOT at the website http://www.analemma.com/. Several equations have been developed for approximating the EOT. One, given by W. Stine and R. Harrigan [6] and attributed by them to L. Lamm, [2] is:

$$EOT \approx \sum_{k=0}^{5}\left[A_k \cos\left(2\pi\frac{kN}{365.25} \right) + B_k \sin\left(2\pi\frac{kN}{365.25} \right) \right], \qquad (24)$$

in minutes, where N is the day number ($N =1$ on Jan. 1, $N =2$ on Jan. 2, etc.) , and the coefficients, A_k and B_k , are given in the following table.

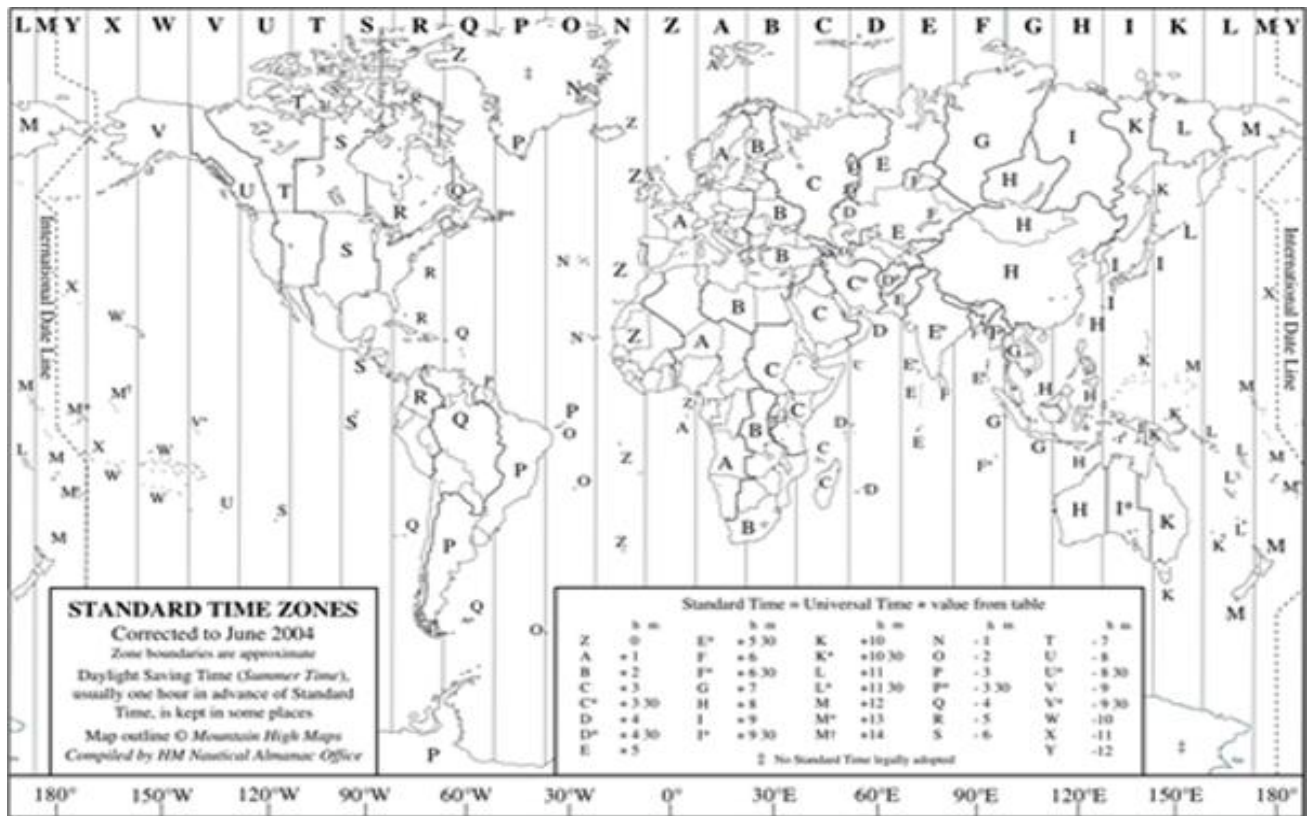| k	A_k	B_k
0	1.252×10^{-2}	$--$
1	5.572×10^{-1}	-7.337
2	-3.135	-9.419
3	-7.846×10^{-2}	-3.096×10^{-1}
4	-1.312×10^{-1}	-1.790×10^{-1}
5	-9.060×10^{-3}	-1.408×10^{-2}

Figure 17 World map of time zones
(from http://aa.usno.navy.mil/faq/docs/world_tzones.html)

J. Meeus [3, Ch. 28] gives an accurate expression for the EOT as it depends on the date and time. One way to display the EOT is by its graph in Figure 18. Another common way is the curve shown in Figure 19 of the solar declination of Figure 14 versus the EOT of Figure 18 with the day number, N, as a parameter along the curve; in Figure 19, the day numbers 1, 32, 60, etc. are indicated and labeled with their corresponding dates Jan. 1, Feb. 1, Mar. 1, etc.
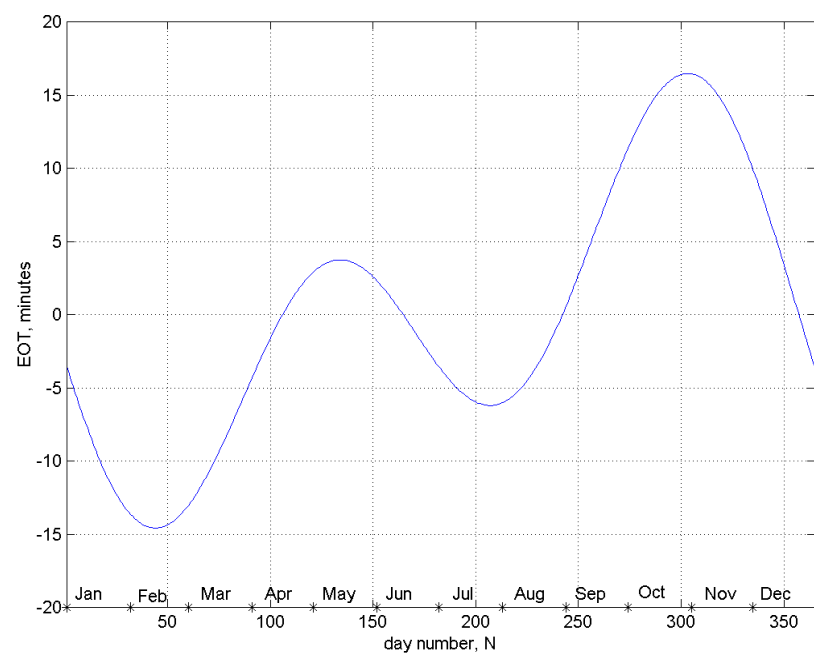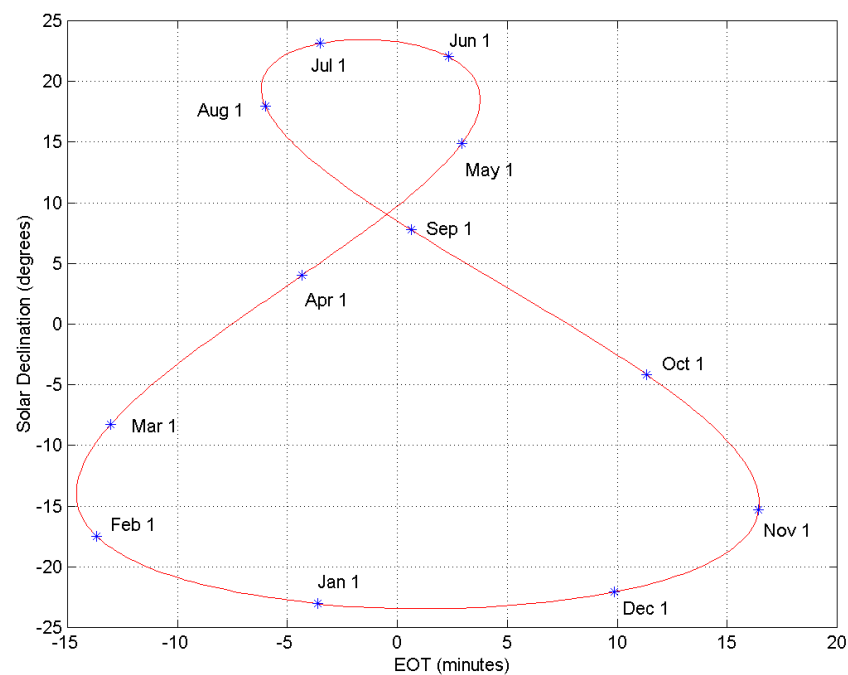
Figure 18 Equation of Time

Figure 19 Solar Declination and the Equation of TIme

The curve in Figure 19 is called the *solar analemma*. It models the sun's apparent position at a given time instant each day over the span of a year. For example, Bjarne H. Madsen made the composite picture of the sun in Figure 20. Using a permanently mounted camera, he acquired photographic exposures of the sun at the same time instant on 44 days between November 2000 and October 2001. These were combined with one foreground photograph to produce the composite picture. See his website at

http://home.worldonline.dk/bhm/analemma.htm.

Additional pictures of the solar analemma and some further explanations can be seen at the following websites:

http://www.perseus.gr/Astro-Solar-Analemma.htm,
http://www.analemma.de/english/analem.html,

and

http://antwrp.gsfc.nasa.gov/apod/ap020709.html.
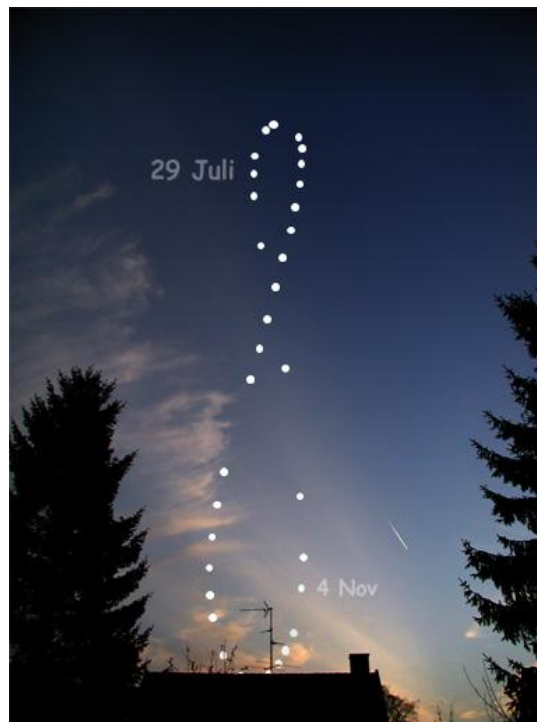

Figure 20 Solar Analemma (photo by B. H. Madsen)

Either Figure 18 or Figure 19 is used to determine the value of the equation-of-time for a particular date of interest. Using (21), this value is subtracted from the time indicated by the sundial in order to obtain clock time.

IV. Analemmatic Sundial Design

An analemmatic sundial consists of two parts. One is an ellipse along which there are time marks. The ellipse is oriented so that its minor axis is in the true, not magnetic, north-south direction for the location of the sundial, and the major axis is perpendicular to that in the east-west direction. The center of the ellipse, at the crossing of the axes, is placed so that the completed sundial is in the desired location. The other part of the analemmatic sundial is a line along which are date marks. This line is in the north-south direction on the minor axis of the ellipse. A shadow caster, typically a person, is placed or stands on a date mark. The shadow of the caster at a certain time on that date then falls towards the appropriate time mark.

Several steps are required in the design of the sundial.

- First, the desired location of the sundial must be identified and made flat and horizontal. A design is still possible if the location is not made to lie in a horizontal plane but will require additional considerations.
- Next, the true north-south direction must be determined, which can be accomplished in a variety of ways, some of which are outlined below. Draw a north-south line once the direction is determined. Then, determine and draw an east-west line passing through the desired center location of the sundial.
- The ellipse defining the analemmatic sundial can now be drawn and the time marks placed on it, as discussed below.
- Finally, the locations of the date marks are determined and the marks placed along the north-south line.

These steps complete the mathematical portion of the design. There are still two more important steps.

- One is the design of the artistic or esthetic aspects of the sundial. Here, there is great freedom and opportunity for creative imagination. The sundial can be quite simple if desired, or it can be elaborate.
- Lastly, the sundial must be constructed, so the appropriate materials need to be selected and construction methods adopted to use those materials. It can be constructed as a temporary project, or it can be one intended to be in place for an extended time.

Finally, when the construction is complete, it's time to try the sundial on a sunny day, take some pictures, and have some fun with the new time piece.

A. Determining the true North-South Direction

R. Rohr [4] and A. Waugh [7] describe several methods for determining the true north-south direction at a given location. A compass can be used, but it will indicate magnetic north. A correction factor is needed to determine true north from magnetic north. Such factors are available on some maps. Another approach is to observe and mark the direction of the pole star, Polaris. Perhaps the easiest method begins with drawing a circle around the center location where the analemmatic sundial is to be placed. Then, erect a narrow rod vertically at the center point of the circle. Put two marks on the circle, one where the tip of the shadow of the vertical rod crosses the circle before noon and the other where it crosses after noon. Draw a line between these two

marks, and construct its perpendicular bisector. This bisector will be in the true north-south direction.

B. Design of the Time Marks

Shown in Figure 21 is an ellipse with its axes aligned to the north-south and east-west directions of the place where the analemmatic sundial is to be placed. The semi-major and semi-minor axes have sizes M and m, respectively. Suppose that at a certain time, the azimuth
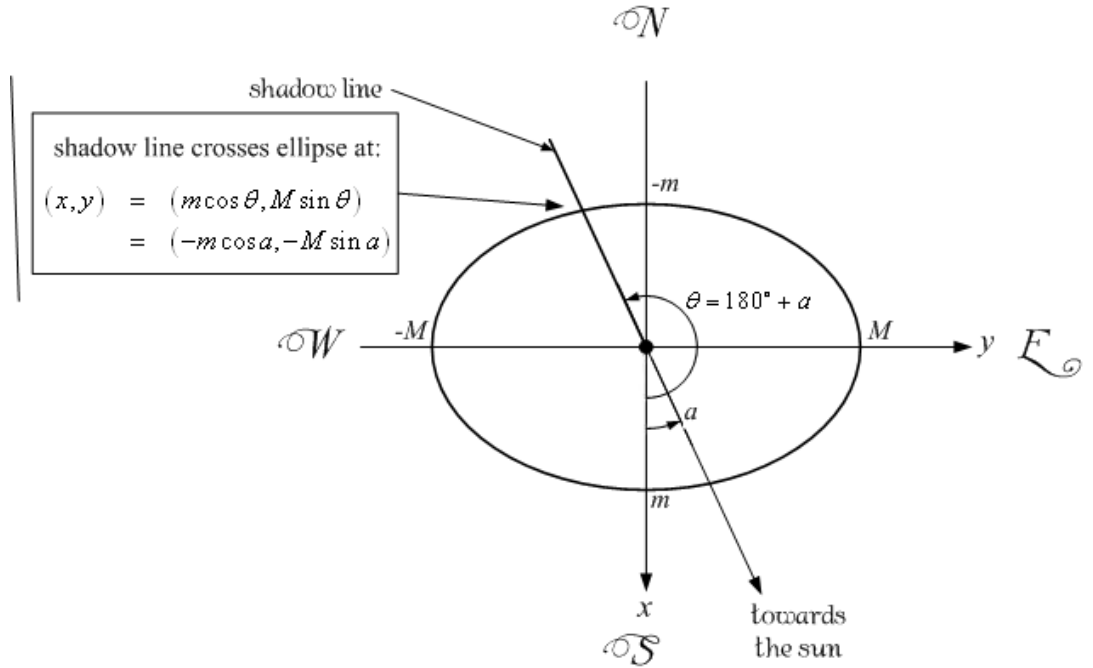
Figure 21 Design of the Time Marks

angle of the sun is a degrees. A vertical rod placed at the origin will cast a shadow at the angle $\theta = 90° + a$ in the opposite direction from the sun. From (2), the location of all points on the ellipse satisfy $(x, y) = (m\cos\theta, M\sin\theta)$ for any choice of θ. In particular, choosing $\theta = 180° + a$ gives the location of the point on the ellipse that is intersected by the line of the shadow cast by the vertical rod placed at the origin when the azimuth angle of the sun is $a°$. This intercept is at $(x, y) = (-m\cos a, -M\sin a)$.

The time marks on the ellipse are located on an equinoctial day. On such a day, the declination of the sun, δ, is zero. From (19), the azimuth angle of the sun then satisfies:

$$\tan a = \frac{\sin\tau}{\cos\tau\sin\phi}, \tag{25}$$

26

where τ is the solar hour angle from the local meridian, and ϕ is the latitude of the location where the dial is to be placed. We wish to mark the ellipse with the time τ when the azimuth of the sun is at this hour angle. If the dimensions of the ellipse are selected so that the minor and major axis dimensions satisfy $m = M \sin\phi$, the intercept is located on the ellipse at

$$(x, y) = (-M \sin\phi \cos\tau, -M \sin\tau), \tag{26}$$

and (25) is satisfied. The intercept point on the ellipse in Figure 21 can then be marked as the solar time τ hours. Various times can be marked by making various choices for τ.

1. Sunrise and Sunset

There is obviously no need to place time marks on the ellipse for times between sunset and sunrise. Equation (18) can be used to determine the times of sunrise and sunset. These can then be used to limit the ellipse of an analemmatic sundial to times of sunshine. Since the elevation angle of the sun is 0° at sunrise and sunset, setting e = 0 in (18) yields

$$\cos\delta \cos\tau_s \cos\phi + \sin\delta \sin\phi = 0, \tag{27}$$

where τ_s is the time of sunset or sunrise. Thus, the times of sunset and sunrise satisfy:

$$\tau_s = \arccos\left(-\tan\delta \tan\phi\right) \tag{28}$$

degrees, or

$$\tau_s = \frac{1}{15}\arccos\left(-\tan\delta \tan\phi\right) \tag{29}$$

hours.

Let a_s be the azimuth angle of the sun at sunset or sunrise. Then, from (15), with e = 0°, and (28)

$$
\begin{aligned}
\cos a_s &= \cos\delta \cos\tau_s \sin\phi - \sin\delta \cos\phi \\
&= -\cos\delta \tan\delta \tan\phi \sin\phi - \sin\delta \cos\phi \\
&= -\frac{\sin\delta}{\cos\phi}
\end{aligned}
\tag{30}
$$

Thus, the azimuth angles of the sun at sunset and sunrise satisfy:

$$a_s = \arccos\left(-\frac{\sin\delta}{\cos\phi}\right). \tag{31}$$

Note that the times of sunrise and sunset occur at an equal number of hours before and after solar noon, respectively. Similarly, the azimuth angles of sunrise and sunset equal in magnitude from south.

For example, on an equinoctial day, when the solar declination is zero, $\delta = 0$, equation (28) indicates that sunrise and sunset occur at an hour angle of ±90° or, from (29), ±6 hours. Similarly, (31) indicates that the azimuth angles of the rising and setting sun are ±90°. Thus, the sun rises at 6 hours before local solar noon (i.e., 6:00 hours, or 6:00 AM) in the direction that is +90° east of south (i.e., east), and it sets at 6 hours after local solar noon (i.e., 18:00 hours, or 6:00 PM) in the direction that is –90° west of south (i.e., west). If the only consideration were the equinoctial day, the only portion of the ellipse in Figure 21 needed for daylight hours is that above the east-west axis. However, on other days daylight hours may extend beyond the 12 hour day from 6:00 AM to 6:00 PM, requiring that more of the ellipse be retained. In fact, the day with the largest number of daylight hours is the day of the summer solstice when the solar declination is 23.45°. Then, from (29), $t_s = 7.35$ hours, so sunrise occurs at 4:39 AM, and sunset occurs at 7:21 PM, and retaining portions of the ellipse below the east-west axis is desirable.

2 Size of the sundial

The parameter M defining the semi-major axis controls the overall size of the analemmatic sundial. The average height of people who will cast the shadow is one consideration in selecting this sizing parameter. As seen in Figure 15, the length of the shadow on any given day is shortest at noon, when the solar elevation is greatest. The noon shadow will in turn be shortest on a day when the noon sun achieves its highest elevation for the year. This is at the summer solstice, June 21. Let e_{max} denote the maximum solar elevation at the location where the sundial will be placed. From Eq. (18),

$$
\begin{aligned}
e_{max} &= \arcsin\left(\cos 23.45^\circ \cos 0^\circ \cos\phi + \sin 23.45^\circ \sin\phi\right) \\
&\approx \arcsin\left(\cos\left(\phi - 23.45^\circ\right)\right).
\end{aligned}
\tag{32}
$$

The shortest shadow length, l_{min}, is then

$$
l_{min} = h \tan e_{max},
\tag{33}
$$

.

where h is the height of the person casting the shadow. One possible choice for sizing the sundial is to select M so that the minor axis of the sundial, $m = M \sin\phi$, is equal to l_{min},

$$
M = \frac{h \tan e_{max}}{\sin\phi},
\tag{34}
$$

where ϕ is the latitude of the place where the sundial is placed. An average height, $\bar{h}$, will need to be used in (34) because people with various heights will be using the sundial.

3. Time Corrections

Time marks placed on the ellipse of Figure 21 according to (26) indicate mean solar-time at the meridian where the sundial is to be placed. Corrections must be applied to convert this time into the legal time indicated by clocks. The necessary corrections are indicated in (21). These are: a correction to account for the difference in longitude between the standard meridian of the time zone in which the sundial is to be located and the meridian of the sundial itself; the equation-of-time correction, which converts mean solar-time into actual solar-time; and, a one hour correction for daylight savings time if that is in effect. These corrections can be incorporated into the design of the sundial in various ways. The longitude correction can be incorporated into the design by replacing the hour-angle, τ, in (26) by $\tilde{\tau}$, where

$$\tilde{\tau} = \tau + \frac{1}{15} \Delta L^{\circ}, \tag{35}$$

in which

$$\Delta L^{\circ} = \text{local-meridian}^{\circ} - \text{standard-meridian}^{\circ}. \tag{36}$$

The resulting time marks are still labeled for the hour angle of τ. The shadow cast by a vertical rod will then be at the 12:00 mark (i.e., $\tau = 0^{\circ}$) when the sun passes through the plane of the standard meridian. If desired, the correction for daylight savings time can be incorporated into the design by labeling time marks with two times, one for standard time and the other for daylight time. Reading time on the sundial then requires that the user know whether or not daylight savings time is in effect and which of the two labels to use. The equation-of-time correction can be incorporated, if desired, by either displaying a graph of the type in Figure 18 at the dial location or an analemma of the type in Figure 19 along the north-south axis of the sundial at the date marks. The user of the sundial must then incorporate this correction by adding (or subtracting) the correction read from the graph or analemma to the time indicated by the time mark at the shadow line.

C. Design of the Date Marks

Suppose that $m = M \sin \phi$ and that the time marks are placed on the ellipse of Figure 21 as described in the preceding section. On an equinoctial day, the shadow of a vertical rod placed at the origin of the ellipse will then fall on each time mark at the time indicated by that mark. This will not be so on other days. However, there is a date-dependent position along the north-south axis where the rod can be placed so that the shadow does intersect the time marks at the appropriate instant on a given date. This position is found by consideration of Figure 22. A vertical rod placed along the north-south axis at $x = d$ casts a shadow towards
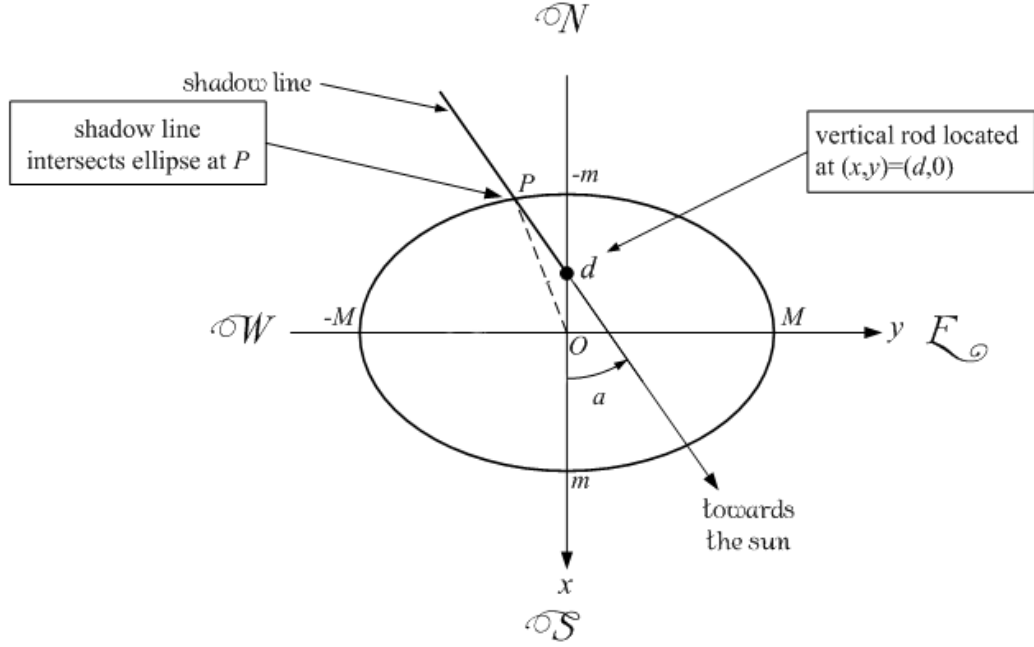
Figure 22 Vertical rod at date mark (x,y)=(d,0) casts a shadow towards point *P* on the ellipse when the sun is at azimuth angle *a*

the time mark, τ, at point P on the ellipse with semi-major axis M and semi-minor axis $m = M \sin \phi$. The equation for the shadow line is

$$x = -y \cot a + d = -y \frac{\cos \tau \sin \phi - \tan \delta \cos \phi}{\sin \tau} + d, \qquad (37)$$

where the second equality is obtained using (19). Now consider the dashed line in Figure 22 from the origin to the point P. This would be the shadow line of the rod if it were placed at the origin on an equinoctial day when $\delta = 0$. The equation for this line is

$$x = -y \frac{\cos \tau \sin \phi}{\sin \tau} \qquad (38)$$

From (25). The location of P on the ellipse can be determined by substituting this expression for x into the equation of the ellipse

$$\frac{x^2}{M^2 \sin^2 \phi} + \frac{y^2}{M^2} = 1 \qquad (39)$$

to get

$$\frac{y^2}{M^2} \left(\frac{\cot^2 \tau \sin^2 \phi}{\sin^2 \phi} + 1 \right) = \frac{y^2}{M^2 \sin^2 \tau} = 1. \qquad (40)$$

Therefore, the *y*-coordinate of *P* is

$$y_P = M \sin \tau, \tag{41}$$

and from (38), the *x*-coordinate is

$$x_P = -M \cos \tau \sin \phi. \tag{42}$$

The choice of *d* so that the shadow line of the rod at $(d, 0)$ passes through *P* is obtained from (37) by substituting $(x, y) = (x_P, y_P)$ and using (41) and (42) to get:

$$-M \cos \tau \sin \phi = -M \sin \tau \left(\frac{\cos \tau \sin \phi - \tan \delta \cos \phi}{\sin \tau} \right) + d, \tag{43}$$

which yields the following expression for the date mark:

$$d = -M \tan \delta \cos \phi. \tag{44}$$

The date determines the solar declination, δ, and the location of the dial determines the latitude ϕ. The location determined by (44) is then labeled with the date. Labels for any dates selected can be placed, but the common choices are the first day of each month. If desired, the date of a special occasion, such as a birthday, wedding anniversary, or holiday, could be selected.

D. Summary of Analemmatic Sundial Design

The steps to design an analemmatic sundial to be located on a flat, horizontal surface at a place with latitude ϕ are:

- Determine and lay out north-south and east-west oriented lines, with their intersection at the center of the site where the sundial is to be placed. See Sec. IV.A for determining the true north-south direction.
- Select a scale *M* for the sundial. See Sec. IV.B.2 for guidelines on doing this.
- Determine and lay out the ellipse of the sundial and time marks along the ellipse. The property (4) of an ellipse provides one way to lay out the ellipse, as discussed below in Sec. VI. Equation (26) is used for locating the time marks. The extent of the ellipse and time marks need only include daylight hours. See Sec. IV.B.1 for determining the times of sunrise and sunset and hence the span of daylight hours. The time marks can be placed to include a time correction for the longitude of the location where the sundial is to be placed. This correction is discussed in Sec. IV.B.3.
- Determine and lay out the date marks along the north-south axis of the ellipse. The location of a date mark is determined by first determining the declination of the sun for the date, for example by using equation (20), and then using equation (44) of Sec. IV.C.

- Choose the artistic aspects of the sundial. Consider including sundial decorations; see Sec. V below for some discussion of this.
- Choose the materials and methods for constructing the sundial. See Sec. VI for some discussion of this.

E. Example: A Design for St. Louis, MO

As an example, consider the design of an analemmatic sundial for St. Louis, MO. Assume that $M = 1$ for the design. All dimensions in the dial can then be scaled for other choices of M. It is desired to place time marks at all daylight hours. The latitude of St. Louis, MO, is 38.6 degrees. The ellipse with time marks has a major axis of dimension $M = 1$ and is aligned with the east-west direction at the location of the sundial. The minor axis has dimension $m = M \sin \phi = \sin 38.6° = 0.624$ and is aligned with the north-south direction. The ellipse is drawn in Figure 23.

Hourly time marks 5:00, 6:00, $\cdots$, 11:00, 12:00, 13:00, $\cdots$, 18:00, and 19:00 are placed on the ellipse at coordinates $(x, y) = \left(-\sin 38.6° \cos \tau, -\sin \tau\right)$ for the corresponding hour angles τ equal to $-105°$, $-90°$, $\cdots$, $-15°$, $0°$, $15°$, $\cdots$, $90°$, and $105°$, respectively. These coordinates are given in the following table, and the time marks are indicated on the ellipse in Figure 23. They were determined using (26) with $M = 1$.

| time, τ | | time mark coordinates | |
hours	degrees	x	y
5:00	-105	+0.162	+0.966
6:00	-90	0.000	+1.000
7:00	-75	-0.162	+0.966
8:00	-60	0.312	+0.866
9:00	-45	-0.441	+0.707
10:00	-30	-0.540	+0.500
11:00	-15	-0.603	+0.259
12:00	0	-0.624	0.000
13:00	15	-0.603	-0.259
14:00	30	-0.540	-0.500
15:00	45	-0.441	-0.707
16:00	60	-0.312	-0.866
17:00	75	-0.162	-0.966
18:00	90	-0.000	-1.000
19:00	105	+0.162	-0.966

Fourteen date marks are placed along the north-south axis, indicating the first day of each month, the winter solstice December 21, and the summer solstice June 21. The coordinates for

these marks were determined using (20) and (44). They are given in the following table, and the date marks are indicated in Figure 23.

date	day number, N	date mark, d
January 1	1	+0.332
February 1	32	+0.247
March 1	60	+0.114
April 1	91	-0.055
May 1	121	-0.208
June 1	152	-0.316
June 21	172	-0.339
July 1	182	-0.334
August 1	213	-0.253
September 1	244	-0.107
October 1	274	+0.057
November 1	305	+0.214
December 1	335	+0.317
December 21	355	+0.339

The ellipse of the sundial in Figure 23 is limited to the hours between 5:00 and 19:00. This range of hours was selected after examining the solar azimuth angles using (31) at sunrise and sunset.
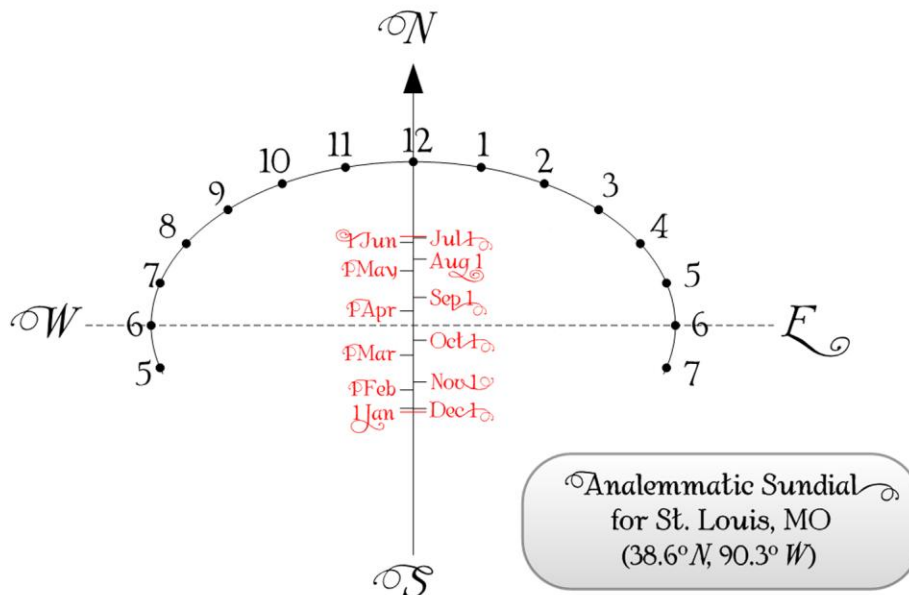

Figure 23 Analemmatic sundial for St. Louis, MO. The solstice dates of June 21 and Dec. 21 are indicated by the red marks on the date line.

Shown in Figure 24 are some shadow lines for the solstice and equinoctial days. The lengths of the shadows of a one-meter rod placed at the appropriate date marks are indicated. For a one meter rod, the length of the shadow is $\cot e$, where e is the elevation angle of the sun, as deter-

mined by (18) for the date of interest. The direction of the shadow is $a + 180°$, where a is the azimuth angle of the sun, as determined by (19) on the date of interest.
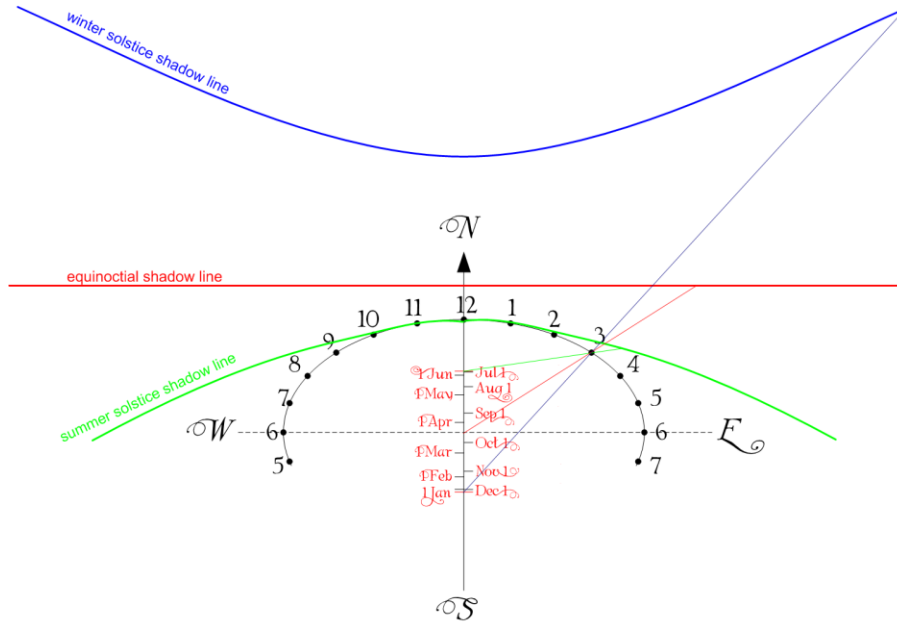

Figure 24 Shadow line lengths for a one-meter rod placed at the winter solstice (Jan. 21, blue), summer solstice (Jun. 21, green), and equinoxes (Mar. 21, Sept. 21, red). The shadow lines at 15:00 hours are shown for each date, and the end points of the shadow lines for each date are shown for all hours.

This sundial indicates local mean-solar-time. Its design does not account for the difference in the longitude of St. Louis, which is 90.3°W and the standard meridian of the time zone in which St. Louis is located, which is 90°. The time correction amounts to $\Delta L° / 15 = +0.02$ hours or +1.2 minutes. For example, when the sundial indicates that the local mean-solar-time is 12:00 noon, the mean-solar-time at the standard meridian of the time zone is 12:01:12. If desired, the dial can be redesigned to indicate mean-solar-time at the standard meridian of the time zone by using $\tilde{\tau} = \tau + 0.02$ in place of τ in (26). The coordinates for the longitudinally corrected time marks are given in the following table, and the resulting sundial is shown in Figure 25. There is only a minor difference between the sundials with and without longitude correction because the longitudes of St. Louis and the standard meridian of the time zone are close to one another. Times during periods when daylight savings time is in effect are indicated in red. A graph of the equation-of-time is included so that clock time can be determined using (21).

time		time mark coordinates	
τ , hours	$\tilde{\tau}$, degrees	x	y
5:00	-104.7	+0.158	+0.967
6:00	-89.7	-0.003	+1.000
7:00	-74.7	-0.165	+0.965
8:00	-59.7	0.315	+0.863
9:00	-44.7	-0.444	+0.703

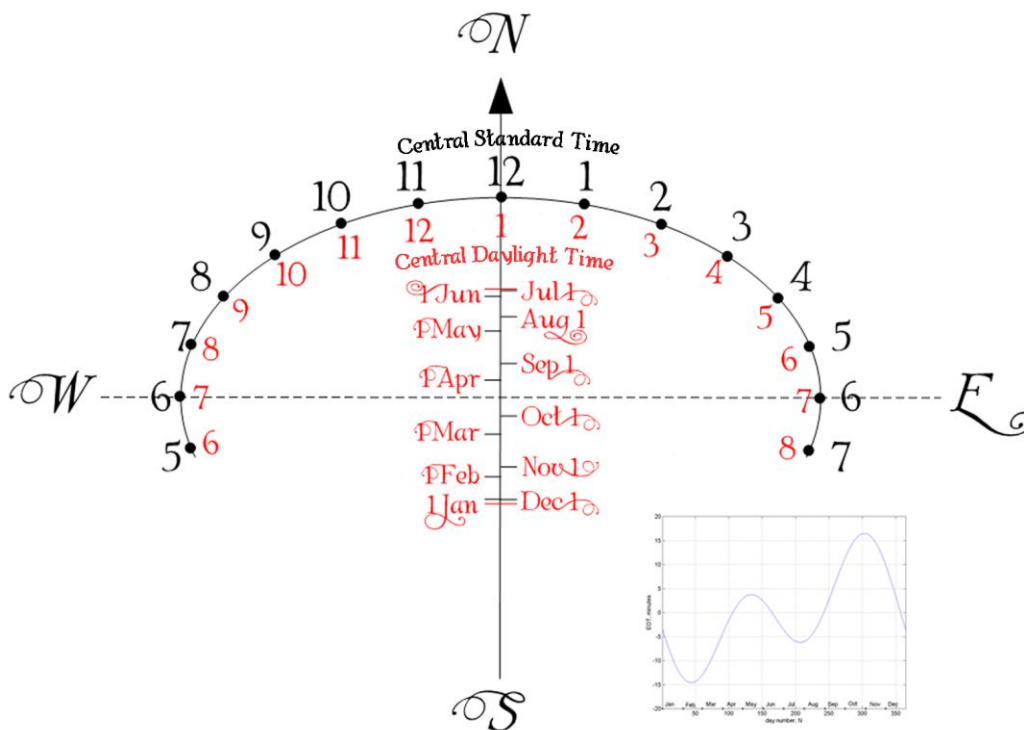|			
10:00	-29.7	-0.542	+0.496
11:00	-14.7	-0.604	+0.254
12:00	0.3	-0.624	-0.005
13:00	15.3	-0.603	-0.264
14:00	30.3	-0.539	-0.505
15:00	45.3	-0.439	-0.711
16:00	60.3	-0.309	-0.869
17:00	75.3	-0.158	-0.967
18:00	90.3	+0.003	-1.000
19:00	105.3	+0.165	-0.965

Figure 25 Analemmatic sundial for St. Louis, MO, corrected for longitude and daylight savings time.

V. Sundial Decorations

Various decorations, also called *furniture*, are commonly included on a sundial. These include a graph of the equation-of-time or an analemma so that clock time can be determined from the time indicated by the sundial. The graph of the EOT is often included on a plaque posted at or near the sundial. The analemma is usually included on the sundial itself along with the date marks on the north-south axis of the sundial. One caution in including the analemma in this way is that users of the sundial may be inclined to stand on the date mark on the analemma rather than the date mark on the north-south axis, which leads to incorrect time readings.

Other forms of furniture include mottos, emblems, date lines, and solar-event lines. Mottos often refer to time in some way. There are many examples, such as:

Make every hour count

> Waste not time
> Time waits for no one
> Sunny hours are happy hours

Lists of mottos that have been used are readily available; see, for example, http://www.sundials.co.uk/mottoes.htm. An emblem may be included on a sundial to indicate its location or a special event it commemorates. Figure 24 displays lines for the equinoctial and solstice days that indicate the end point of the shadow of a 1 meter rod throughout the course of these days of special solar events. Such a line can be included for any date. For example, the line for a holiday related to the commemoration of the sundial could be included, as could someone's birthday or wedding day. However, such lines may not be desirable for an analemmatic sundial because of the varying heights of its users.

VI. Construction Materials and Methods

There are many options in constructing an analemmatic sundial once its ellipse, time marks, and date marks are designed. There is a wide choice of materials that may be used. The character of the sundial can be greatly affected by inventive, artistic choices in its implementation. The examples in Figure 2 show a small sampling of the variety of possibilities of construction.

All implementations will require the determination of the direction of true north, laying out the axes of the sundial, drawing the ellipse, and placing the time and date marks. A compass alone cannot be used to determine the direction of true north. The deviation between magnetic and true north varies with location. This deviation is indicated on some maps, and a compass can be used if this deviation information is available as well. The direction of true north is easy to measure. One method is to draw a circle with a vertical rod placed at its center. The tip of the shadow of the rod will cross the circle once before noon and once after noon. The perpendicular bisector of the line connecting these two crossover points is the true north-south axis of the sundial. The center of the ellipse that will form the sundial can then be marked along the north-south axis, and the perpendicular line drawn at this center point will be the east-west axis of the ellipse. The length of the semi-major and semi-minor axes of the ellipse, M and $m = M \sin\phi$, can then be marked. Also, the two foci of the ellipse, $f = \pm\sqrt{M^2 - m^2} = \pm M \cos\phi$, can be marked. One approach for laying out the ellipse itself is to use the property of an ellipse in (4). The endpoints of a rope of length $2M$ are secured at the foci of the ellipse. A point P is the ellipse when the rope is stretched taut, as in Figure 4. The ellipse can be marked out by moving the point P to various locations while keeping the rope taut. Time and date marks can be added by using a ruler to measure their coordinates along the north-south and east-west axes.

VII. Discussion and Conclusions

The design of analemmatic sundials is based on predicting solar positions and the properties of ellipses. The properties of ellipses are reviewed in Section II-A. Earth-centered and place-centered coordinate systems are used in predicting solar positions as the earth rotates about its axis while orbiting the sun. These coordinate systems are reviewed in Section II-C. Sundials indicate mean-solar-time, so corrections must be applied to obtain clock time from a sundial reading. These corrections are discussed in Section III. All of these ideas are brought together in the design of analemmatic sundials, as discussed in Section IV.

VIII. Bibliography

The following is a short list from among many publications that deal with predicting solar positions and the theory of sundials.

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2. L. O. Lamm, "A new analytic expression for the Equation of Time," Solar Energy, Vol. 26, p. 465, 1981.

3. J. Meeus, *Astronomical Algorithms*, *Second Edition*, Willmann-Bell, Inc., 1999.

4. R. R. J. Rohr, *Sundials: History, Theory and Practice*, Dover, 1970.

5. F. W. Sawyer III, "Of analemmas, mean time, and the analemmatic sundial," in: *The Analemmatic Sundial Sourcebook*, a festschrift on the tenth anniversary of the North American Sundial Society, 2004.

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7. A. E. Waugh, *Sundials: Their Theory and Construction*, Dover, 1973.

related websites

"Keppel Henge," www.steveirvine.com/sundial.html. (A discussion of constructing an analemmatic sundial is included.)

"Painting a Sunclock or Human Sundial," http://www.sunclocks.com/index.htm.

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"Time zones and 'Z' time (universal time)," http://www.aos.wisc.edu/~hopkins/wx-doc/z-time.html.

North American Sundial Society, http://sundials.org. (Source of information about all things related to sundials.)